

Multiplication law and S transform for non-hermitian random matrices

Z. Burda,* R.A. Janik,[†] and M.A. Nowak[‡]

*Marian Smoluchowski Institute of Physics and Mark Kac Complex Systems Research Center
Jagiellonian University, Reymonta 4, 30-059 Kraków, Poland*

(Dated: January 19, 2013)

We derive a multiplication law for free non-hermitian random matrices allowing for an easy reconstruction of the two-dimensional eigenvalue distribution of the product ensemble from the characteristics of the individual ensembles. We define the corresponding non-hermitian S transform being a natural generalization of the Voiculescu S transform. In addition we extend the classical hermitian S transform approach to deal with the situation when the random matrix ensemble factors have vanishing mean including the case when both of them are centered. We use planar diagrammatic techniques to derive these results.

PACS: 05.40.+j; 05.45+b; 05.70.Fh; 11.15.Pg

Keywords: Non-hermitian random matrix models, Diagrammatic expansion, Products of random matrices.

I. INTRODUCTION

Free random variables [1, 2] play an increasingly important role in mathematics, physics, multivariate statistics and interdisciplinary research [3–10]. The cornerstones of this success are the so-called R and S transforms. The R transform allows one to infer the spectral properties of the sum of random operators, provided the individual spectral measures are known for each of them and they are independent in the noncommutative sense also known as free. The S transform plays a similar role for the multiplication of free random operators. These constructions allow for fast decomposition of several problems for complicated random operators into simple ingredients. Since free random operators have an explicit realization in terms of infinitely large random matrices, the techniques based on the R and S transforms provide a powerful tool to solve technically involved problems in random matrix theory in an easy way when traditional methods break down.

Historically, the R transform was devised for hermitian operators and the S transform for unitary ones. The issue of the generalization of these constructions to other classes of operators was a subject of intensive research during the last two decades. In particular, one of the most challenging problems was the question of the possibility of an extension of the R and S transforms to strictly non-hermitian matrices, which find nowadays vast applications in many fields of research. This problem is also especially interesting as traditional techniques developed for hermitian random matrices generally fail in the non-hermitian case. Some time ago, two of the present authors have extended the *additive* R transform for the non-hermitian ensembles [11, 12]. Similar constructions were also proposed independently in [13, 14], and were soon generalized [15, 16]. The question of defining the *multiplicative* S transform for non-hermitian matrices was however open and frequently doubts were expressed whether such a construction is possible at all. On the other hand several complicated problems involving products of large matrices have been solved using other methods and results were sometimes surprisingly simple [17–22], hinting at the possibility of a hidden mathematical structure.

In this work we demonstrate that such a structure – the non-hermitian S transform – exists and can be used as a powerful algorithm for solving the spectral problems of various products of random matrices. As a byproduct we also generalize the ordinary ‘hermitian’ multiplicative technique to matrix ensembles with vanishing mean which was never done before.

In Section II we outline main results of the paper. In particular we give the multiplication law for free non-hermitian matrices.

In the next two sections, in order to make the paper self-contained, we introduce diagrammatic techniques which will be the main tool for deriving the key results of this paper.

In Section III we very briefly recall the formalism to calculate the eigenvalue densities of large random hermitian matrices in the limit of matrix dimensions $N \rightarrow \infty$. We recall the connection to planar diagrams and use the diagrammatic technique to give a simple proof of the addition law.

In Section IV we repeat the discuss for non-hermitian matrices. We show that the Green’s function and the R transform are given by 2×2 matrices and recall the formalism to handle this case.

* zdzislaw.burda@uj.edu.pl

† romuald@th.if.uj.edu.pl

‡ nowak@th.if.uj.edu.pl

In Section V, which is the main section of this paper, we first rederive the multiplication law for hermitian matrices using diagrammatic arguments and then we generalize the construction to non-hermitian matrices. We discuss the S transform for this case and show that similarly to the nonhermitian versions of the R transform and the Green's function it has a form of a 2×2 matrix.

Finally in Section VI we give examples of application of this law to practical calculations of the eigenvalue density for a product of free matrices. We conclude the paper with a short summary.

II. MAIN RESULTS

In this section we shortly summarize the main results of this paper. The key quantity of interest in random matrix theory is the eigenvalue density, which may be equivalently expressed through the Green's function. The R and S transforms satisfy functional relations with the Green's function and hence their knowledge is equivalent (in the hermitian case) to the knowledge of the eigenvalue density (or more precisely of its moments).

Explicitly, the standard form of the multiplication law of free large hermitian matrices is given in terms of the S transform [1] just through an ordinary product

$$S_{AB}(z) = S_A(z)S_B(z) \quad (1)$$

The S transform is a complex function of a complex variable and it is related to the R transform as follows

$$S(z) = \frac{1}{R(zS(z))} \quad (2)$$

The two relations given above hold only if matrices A and B are not centered: $\langle \text{tr}A \rangle \neq 0$ and $\langle \text{tr}B \rangle \neq 0$. This means the corresponding R transforms may not vanish at the origin of the complex plane $R_A(z=0) \neq 0$ and $R_B(z=0) \neq 0$. If either $R_A(0) = 0$ or $R_B(0) = 0$ but not both, the corresponding S transforms do not exist, but one can still save the multiplication law [18]. The prescription [18] breaks down when both means (i.e. for A and B) ensemble vanish. One of our main new results is that one can still write down a multiplication law in terms of the R transform in that case too, using the the following set of equations

$$\begin{aligned} R_{AB}(z) &= R_A(w)R_B(v) \\ v &= zR_A(w) \\ w &= zR_B(v) \end{aligned} \quad (3)$$

which involves three complex variables z, w, v . One can eliminate w and v for given R_A and R_B to obtain $R_{AB}(z)$. This set is equivalent to the standard equation (1) when the matrices A and B are not centered but it is also valid when either of the two matrices, or even both, are centered, making this a more general formulation. This set of equations is quite handy in practical calculations too. One can use it to directly calculate the R transform of the free product avoiding the determination of all auxiliary functions and the S transform in particular. Another advantage of these equations is that they can be generalized in a natural manner to the case of free multiplication of non-hermitian operators and thus they can be used to determine the eigenvalue distribution of products of non-hermitian matrices taken from independent random ensembles in the large N limit.

Before we write down the corresponding set of equations let us first recall that the Green's function for non-hermitian matrices are conveniently expressed as two-by-two matrices with complex elements [11, 12]. This will be in detail explained in the paper. The R transform in this case is a map of a space of two-by-two complex matrices onto a space of two-by-two complex matrices $\mathcal{G} \rightarrow \mathcal{R}(\mathcal{G})$. In order to distinguish this situation from the hermitian case (3) where functions and their arguments were complex numbers we shall use calligraphic letters to denote the corresponding two-by-two complex matrices. The law of free multiplication for non-hermitian matrices reads

$$\begin{aligned} \mathcal{R}_{AB}(\mathcal{G}) &= [\mathcal{R}_A(\mathcal{G}_B)]^L \cdot [\mathcal{R}_B(\mathcal{G}_A)]^R \\ [\mathcal{G}_A]^R &= \mathcal{G} \cdot [\mathcal{R}_A(\mathcal{G}_B)]^L \\ [\mathcal{G}_B]^L &= [\mathcal{R}_B(\mathcal{G}_A)]^R \cdot \mathcal{G}. \end{aligned} \quad (4)$$

It has almost an identical algebraic structure as (3) except that now all objects are two-by-two matrices and thus the order of multiplications matters. The superscripts R and L outside the square brackets, which were absent in (3), stand for right or left rotations, respectively, of a matrix X in the brackets: $[X]^L = UXU^\dagger$ and $[X]^R = U^\dagger XU$. The matrix U is a unitary diagonal matrix $U = \text{diag}(e^{i\phi/4}, e^{-i\phi/4})$ that depends on the phase ϕ of the complex number $z = |z|e^{i\phi}$ being the argument of the Green function $\mathcal{G} = \mathcal{G}(z, \bar{z})$ containing the information on the spectral

distribution of complex eigenvalues on the complex plane z . Although this set of equations is more complicated than for hermitian matrices (3) it also gives a direct, practical way of determining the Green's function \mathcal{G} of the product of random matrices A and B . We will illustrate this by an explicit examples towards the end of the paper. We will also introduce the S transform for non-hermitian matrices and use it to rewrite the set of equations (4), however we think that from the operational point of view equations (4) are more convenient.

III. HERMITIAN MATRICES

A. Preliminaries

We are interested in finding the distribution of eigenvalues λ_i , in the limit when N (the size of the matrix H) is infinite. The average spectral distribution reads

$$\rho(\lambda) = \lim_{N \rightarrow \infty} \frac{1}{N} \left\langle \sum_{i=1}^N \delta(\lambda - \lambda_i) \right\rangle \quad (5)$$

where λ_i are eigenvalues of a random hermitian matrix H and brackets $\langle \dots \rangle$ denote averaging over a given ensemble of $N \times N$ random hermitian matrices generated with the probability

$$P(H) \propto e^{-N \text{Tr} V(H)}. \quad (6)$$

For hermitian matrices eigenvalues λ_i 's lie on the real axis. It is convenient to introduce a complex-valued resolvent (Green's function)

$$G(z) = \lim_{N \rightarrow \infty} \frac{1}{N} \left\langle \text{Tr} \frac{1}{z \mathbb{1} - H} \right\rangle. \quad (7)$$

from which one can reconstruct the spectral density function (5)

$$\rho(\lambda) = \frac{1}{2\pi i} \lim_{\epsilon \rightarrow 0^+} (G(\lambda - i\epsilon) - G(\lambda + i\epsilon)). \quad (8)$$

using the well-known formula $\frac{1}{\lambda \pm i0^+} = \text{P.V.} \frac{1}{\lambda} \mp i\pi\delta(\lambda)$. The symbol $\mathbb{1}$ will be used throughout the paper to denote identity matrices of different size. Here it was an N -by- N identity matrix. The Green's function is a generating function for spectral moments $\mu_n = \lim_{N \rightarrow \infty} \frac{1}{N} \langle \text{Tr} H^n \rangle$

$$G(z) = \sum_{n=0}^{\infty} \frac{\mu_n}{z^{n+1}} \quad (9)$$

with $\mu_0 = 1$, as follows from the $1/z$ -expansion of (7). Another fundamental quantity is the "self-energy" $\Sigma = \Sigma(z)$ defined as

$$G(z) = \frac{1}{z - \Sigma(z)}. \quad (10)$$

It is related to the Green's function by an independent equation

$$\Sigma(z) = R(G(z)), \quad (11)$$

where the function

$$R(z) = \sum_{n=1}^{\infty} \kappa_n z^{n-1} \quad (12)$$

is the generating function for planar connected moments $\kappa_n = \lim_{N \rightarrow \infty} \frac{1}{N} \langle \langle \text{Tr} H^n \rangle \rangle$ called free cumulants and denoted by double brackets. This function is usually referred to as the R transform. Its form can be deduced from the integration measure (6). The difference between the planar connected moments (free cumulants) κ_n in (12) and the spectral moments μ_n (9) will be explained in the next section where a diagrammatic interpretation of these equations will be discussed.

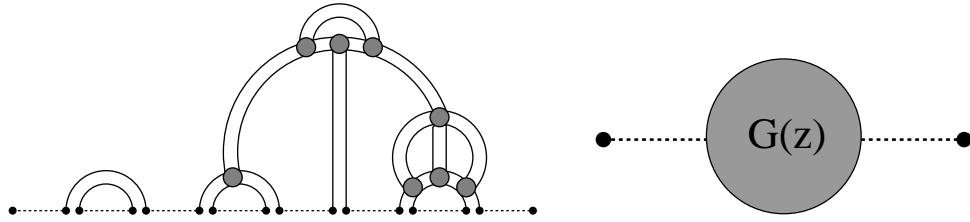


FIG. 1. (Left) An example of a diagram contributing to the generating function $G_{ij}(z)$. Two end-points should be labeled by indices ij . Each horizontal dashed line corresponds to $\frac{1}{z}\delta_{ab}$ while a double line represents the expectation value (the propagator) $\langle H_{ab}H_{cd} \rangle_0 = \frac{1}{Ng_2}\delta_{ad}\delta_{bc}$. Since all lines in the diagram are proportional to the delta function this equation reduces to a scalar equation for $G(z)$. Each horizontal dashed line corresponds after this reduction to $\frac{1}{z}$, each double line to $\frac{1}{g_2}$, each vertex to g_n . The shown diagram contributes to the seventh moment $\frac{1}{N}\langle \text{Tr}H^7 \rangle$ which of order $\frac{1}{z^8}$ in the series expansion (9) since it has eight horizontal lines. The diagram contains seven cubic vertices g_3^7 and one quartic vertex g_4 that are generated by the perturbative expansion of the residual part of (15). Each pair of dots on the horizontal line corresponds to a factor H_{ab} inside the average $\langle \text{Tr}H^7 \rangle_0$. (Right) The graphical notation for the generating function $G(z)$. It generates diagrams having two end-points which include for example the one shown on the left.

The relation between the generating function for spectral moments $G(z)$ and the generating function for connected moments $R(z)$ can be made explicit if one eliminates Σ from (10) and (11). This yields a relation

$$G(z) = \frac{1}{z - R(G(z))} \quad (13)$$

which is equivalent to

$$G\left(R(z) + \frac{1}{z}\right) = z. \quad (14)$$

One can use these relations to determine $G(z)$ for given $R(z)$ or vice versa. To give an example, consider the simplest case of a random matrix from the Gaussian Unitary Ensemble (GUE). In this case the only non-vanishing cumulant is κ_2 . Without loss of generality we can choose $\kappa_2 = 1$, so that $R(z) = z$. Using (13) we have $G(z) = 1/(z - G(z))$. The last equation can be easily solved for $G(z)$ and the solution can be used to calculate the spectral density (8). One recovers the Wigner's semicircle $\rho(\lambda) = \frac{1}{2\pi}\sqrt{4 - \lambda^2}$ [23].

B. Planar diagrams

One can calculate (9) by Gaussian perturbation theory. One does it by splitting the integration measure (6) into a Gaussian part and a residual part

$$P(H) = \mathcal{N}^{-1} e^{-N\frac{g_2}{2}\text{Tr}H^2} e^{-N\sum_{n\neq 2}\frac{g_n}{n}\text{Tr}H^n} = P_0(H)e^{-N\sum_{n\neq 2}\frac{g_n}{n}\text{Tr}H^n} \quad (15)$$

The Gaussian part $P_0(H)$ is then used to calculate averages $\langle \dots \rangle_0$ while the remaining expression is left inside the brackets and is averaged with respect to P_0 . The constant \mathcal{N} is an overall normalization. This non-Gaussian part is perturbatively expanded in g_n , so effectively one has to calculate averages of various powers of H with respect to the Gaussian measure. Each term in this expansion has a graphical representation, similar to Feynman diagrams known from quantum field theory (see figure 1). For example, single horizontal lines represent contributions from the factors $\frac{1}{z}\mathbb{1}$ in (9). In the large N limit only planar diagrams contribute to $G(z)$, since all others are suppressed by $O(1/N)$ factors (note that each closed line generates a factor N coming from contraction of indices $\delta_{ii} = N$). Thus the calculation of $G(z)$ amounts to summing all (infinitely many) contributions from planar diagrams with two endpoints as shown in figure (1). Actually in the most general case one should rather consider a matrix form of the Green's function $\mathbb{G} = (G_{ij}(z))$ where i and j are indices of two end-points $i = 1, \dots, N$, $j = 1, \dots, N$ (see figure 1) and calculate the scalar function (7) afterwards as the normalized trace $G(z) = \frac{1}{N}\text{Tr}\mathbb{G}(z)$. Also the self-energy equation (10) should formally be written in a matrix form. However in our case all generating matrices are proportional to Kronecker delta functions $z_{ij} = z\delta_{ij}$, $G_{ij}(z) = G(z)\delta_{ij}$, $\Sigma_{ij}(z) = \Sigma(z)\delta_{ij}$, $\langle H_{ij}H_{kl} \rangle_0 \sim \delta_{il}\delta_{jk}$ so all equations like (10) and (11) reduce to scalar equations for the coefficients multiplying the delta functions.

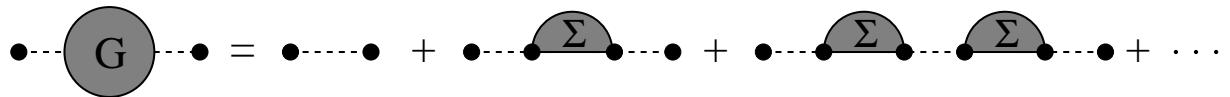


FIG. 2. Diagrams $G(z)$ can be obtained from one-line-irreducible diagrams $\Sigma(z)$ (see figure 3) by joining them one after another.

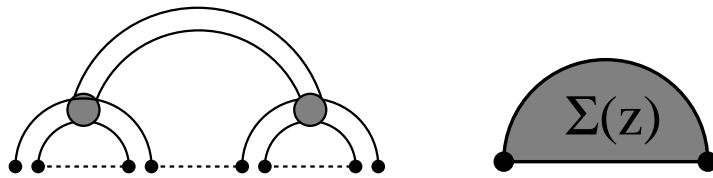


FIG. 3. (Right) An example of an one-line-irreducible diagram. (Left) The graphical notation for the generating function $\Sigma(z)$ of one-line-irreducible diagrams.

A graphical interpretation of equation (10) becomes clear if one rewrites it as an infinite geometric series

$$G(z) = \frac{1}{z} + \frac{1}{z}\Sigma(z)\frac{1}{z} + \frac{1}{z}\Sigma(z)\frac{1}{z}\Sigma(z)\frac{1}{z} + \dots . \quad (16)$$

which can be seen in figure 2. This figure tells us that all diagrams in $G(z)$ can be constructed by lining up one-line-irreducible diagrams one after another. An example of such a one-line-irreducible diagram contributing to $\Sigma(z)$ is shown in figure 3. Such diagrams are characterized by the property that they cannot be disconnected by cutting one line as opposed to diagrams generated by $G(z)$. Indeed, as one can see in figure 2 a diagram in $G(z)$ can be disconnected by cutting any horizontal line like that between two consecutive Σ 's. The diagrammatic equation in figure 2 can be interpreted as a definition of the generating function $\Sigma(z)$ of one-line-irreducible diagrams.

It turns out that one can write down an independent equation relating $\Sigma(z)$ to $G(z)$. One can namely observe that any one-line-irreducible diagram can be obtained from diagrams generated by $G(z)$ as shown in figure 4 by adding a spider structure making them one-line-irreducible. Each bubble κ_n of the spider with n double legs corresponds to a connected moment (free cumulant) of order n . This equation tells us that

$$\Sigma(z) = \frac{1}{N} \langle \langle \text{Tr} H \rangle \rangle + \frac{1}{N} \langle \langle \text{Tr} H^2 \rangle \rangle G(z) + \frac{1}{N} \langle \langle \text{Tr} H^3 \rangle \rangle G^2(z) + \dots = R(G(z)) \quad (17)$$

The diagrammatic equations in figures 2 and 4 belong to the category of Dyson-Schwinger equations known from quantum field theory. They are equivalent to the equations (10) and (11) discussed in the previous section.

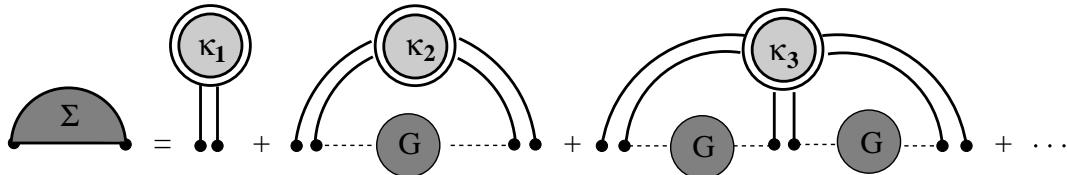


FIG. 4. A one-line-irreducible diagram can be obtained from a one-line-reducible diagram by adding to it a minimal diagrammatic structure complying with the measure (15) which makes it one-line-irreducible. Such a minimal structure is provided by diagrams corresponding to planar connected moments κ_k (free cumulants) (see figure 5) which we indicated by bubbles surrounded by double circles in the figure. This double ring around the bubble is chosen to make it similar to double brackets used in our notation for connected averages. Diagrams in such a bubble are connected. The difference between diagrams corresponding to planar connected moments (cumulants) and spectral moments is explained in figure 5.

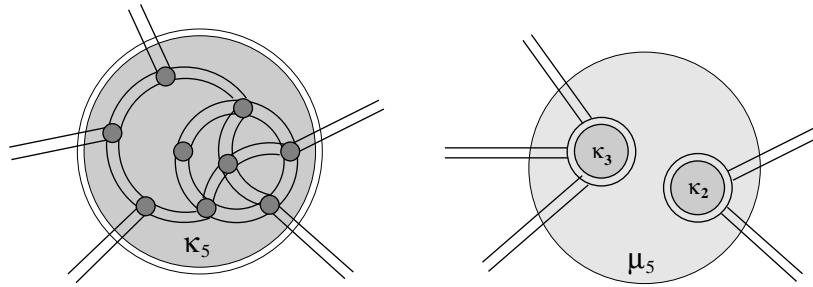


FIG. 5. (Left) An example of a diagram generated by fifth free cumulant $\kappa_5 = \frac{1}{N} \langle \langle \text{Tr} H^5 \rangle \rangle$. All diagrams in the bubble must be connected in contrast to the diagrams generated by spectral moments. (Right) An example of the decomposition of some diagrams generated by the fifth spectral moment $\mu_5 = \frac{1}{N} \langle \text{Tr} H^5 \rangle$ into two connected moments $\kappa_2 \kappa_3$. Some other diagrams in μ_5 can be decomposed into $\kappa_1 \kappa_2 \kappa_2$ or any other combination of cumulants as long as the number of external legs is five. Only a small subset of diagrams in μ_5 corresponds to those of κ_5 .

C. Addition law: R transform

The R transform [1] is important because it allows one to concisely write down a law of addition of (free) independent large matrices. Consider first a factorized measure for two large matrices A, B in the limit $N \rightarrow \infty$

$$P(A, B) = P_A(A)P_B(B) \quad (18)$$

where $P_A(A) \sim \exp -N \text{Tr} V_A(A)$ and $P_B(B) \sim \exp -N \text{Tr} V_B(B)$. Then consider a matrix $H = A + B$. The law of addition tells us how to calculate the spectral density of H for given spectral densities of A and B .

The idea is based on the observation that connected planar moments (free cumulants) of the sum $H = A + B$ split into two independent parts

$$\frac{1}{N} \langle \langle \text{Tr}(A + B)^k \rangle \rangle = \frac{1}{N} \langle \langle \text{Tr} A^k \rangle \rangle + \frac{1}{N} \langle \langle \text{Tr} B^k \rangle \rangle. \quad (19)$$

The reason for this separation of connected moments can be easily understood in terms of Feynman diagrams. All mixed connected moments $\langle \langle \text{Tr} A^a B^b A^c B^d \dots \rangle \rangle$ disappear just because there is no direct line in a connected diagram between a vertex of type A and B since the AB -propagator is zero $\langle A_{ij} B_{kl} \rangle_0 = 0$. The crossed pairs of double lines corresponding to A and B vanish in the large N limit, since they represent non-planar contribution. So all external lines of a bubble generated by k -th cumulant correspond either to A or to B . In other words free cumulants fulfill a simple equation

$$\kappa_{A+B,n} = \kappa_{A,n} + \kappa_{B,n} \quad (20)$$

and thus

$$R_{A+B}(z) = R_A(z) + R_B(z). \quad (21)$$

The argument given above is equivalent to a reasoning based on non-crossing partitions used to prove this law in [2]. The law of free addition (21) is also sufficient to calculate spectral moments of the free sum $\mu_n = \frac{1}{N} \langle \text{Tr} H^n \rangle = \frac{1}{N} \langle \text{Tr} (A + B)^n \rangle$ if one knows the spectral moments of A and B . The recipe follows from the relations (13) and (14):

1. Using (14) calculate $R_A(z)$ for given $G_A(z)$ and $R_B(z)$ for $G_B(z)$.
2. Construct the R transform $R_{A+B}(z)$ for the sum using the addition law (21).
3. Calculate $G_{A+B}(z)$ for $R_{A+B}(z)$ using (14) and calculate spectral moments $\langle \text{Tr} (A + B)^n \rangle$ and the spectral density of $A + B$ using (8).

IV. NON-HERMITIAN RANDOM MATRICES

A. Preliminaries

We now briefly recall how to calculate the spectral density of non-hermitian random matrices using generalized Green's functions [12]. The crucial difference between the hermitian and non-hermitian case comes from the fact

that in non-hermitian random matrix models eigenvalues do not lie on the real axis. In the large N limit they may accumulate in *two-dimensional* domains in the complex plane and the corresponding eigenvalue density

$$\rho(z, \bar{z}) = \lim_{N \rightarrow \infty} \frac{1}{N} \left\langle \sum_i \delta^{(2)}(z - \lambda_i) \right\rangle \quad (22)$$

may become a continuous function with an extended support in the complex plane. In particular, in stark contrast to the hermitian case, the moments $\mu_n = \frac{1}{N} \langle \text{Tr} X^n \rangle$ no longer determine the eigenvalue density. If one wants to apply the Green's function formalism for (22) one has to find a representation of the two-dimensional delta function and not as in the previous section of one dimensional one (5). A natural candidate is

$$\delta^{(2)}(z - \lambda_i) = \frac{1}{\pi} \lim_{\epsilon \rightarrow 0} \frac{\epsilon^2}{(\epsilon^2 + |z - \lambda_i|^2)^2} . \quad (23)$$

With help of this representation one can write

$$\rho(z, \bar{z}) = \lim_{\epsilon \rightarrow 0} \lim_{N \rightarrow \infty} \left\langle \frac{1}{N} \sum_{i=1}^N \frac{\epsilon^2}{(\epsilon^2 + |z - \lambda_i|^2)^2} \right\rangle \quad (24)$$

or

$$\rho(z, \bar{z}) = \frac{1}{\pi} \frac{\partial^2 F(z, \bar{z})}{\partial z \partial \bar{z}} \quad (25)$$

where

$$F(z, \bar{z}) = \lim_{\epsilon \rightarrow 0} \lim_{N \rightarrow \infty} \left\langle \frac{1}{N} \sum_{i=1}^N \ln(|z - \lambda_i|^2 + \epsilon^2) \right\rangle \quad (26)$$

or equivalently

$$F(z, \bar{z}) = \lim_{\epsilon \rightarrow 0} \lim_{N \rightarrow \infty} \left\langle \frac{1}{N} \text{Tr} \ln((z \mathbb{1} - X)(\bar{z} \mathbb{1} - X^\dagger) + \epsilon^2 \mathbb{1}) \right\rangle . \quad (27)$$

One can interpret (24) as a Poisson equation for electrostatics where $\rho(z, \bar{z})$ is a two-dimensional charge distribution and $F(z, \bar{z})$ is a electrostatic potential [24–26]. One can further exploit the electrostatic analogy by introducing the corresponding electric field which is equal to the Green's function

$$G(z, \bar{z}) \equiv \frac{\partial F}{\partial z} = \lim_{\epsilon \rightarrow 0} \lim_{N \rightarrow \infty} \left\langle \frac{1}{N} \text{Tr} \frac{\bar{z} \mathbb{1} - X^\dagger}{(\bar{z} \mathbb{1} - X^\dagger)(z \mathbb{1} - X) + \epsilon^2 \mathbb{1}} \right\rangle . \quad (28)$$

up to a coefficient. F is a real function on the complex plane, so it is a scalar field from the point of view of two-dimensional electrodynamics while G is a complex function and a vector field, respectively. The Poisson equation can be rewritten as a Gauss law in two-dimensions

$$\rho(z, \bar{z}) = \frac{1}{\pi} \partial_{\bar{z}} G(z, \bar{z}) . \quad (29)$$

In the large N limit when the eigenvalues λ_i of the random matrix coalesce in a certain region of the complex plane, the Green's function $G(z, \bar{z})$ is no longer holomorphic. Actually as one can see from the Gauss law (29) the eigenvalue distribution $\rho(z, \bar{z})$ is related to the non-holomorphic behavior of the Green's function.

Let us make a few general remarks about the way we shall use this electrostatic interpretation. In electrostatics one usually applies the Gauss law to determine the electric field for a given charge density. In our problem we proceed in the opposite direction. We first calculate the Green's function (electric field) and then we use it to determine the eigenvalue density. Secondly, in order to calculate the average (28) one has to take a double limit. It is important to take it in the correct order: first to send N to infinity and only then ϵ to zero, since if one took this limit in the opposite order by first setting $\epsilon = 0$ for a finite matrix, then the expression in the brackets in (28) would reduce to $1/N \text{Tr}(z \mathbb{1} - X)^{-1}$. Finally, whenever we apply generating functions for planar diagrams we can automatically take the limit $\epsilon \rightarrow 0$, which trivially amounts to setting $\epsilon = 0$, since the large N limit ($N \rightarrow \infty$) has already been taken by the planar approximation used to write relations between generating functions for planar diagrams.

Note that the Green's function (28) is a complicated object which does not resemble its hermitian counterpart – in particular we cannot just apply the geometric series expansion that was crucial for calculations in the hermitian case

(9). We can however use a trick, invented in [12], which allows us to apply the geometric series expansion but for an extended $2N \times 2N$ matrix:

$$\mathcal{G}(z, \bar{z}) = \begin{pmatrix} \mathcal{G}_{11} & \mathcal{G}_{1\bar{1}} \\ \mathcal{G}_{\bar{1}1} & \mathcal{G}_{\bar{1}\bar{1}} \end{pmatrix} = \left\langle \frac{1}{N} \text{Tr}_{b2} \begin{pmatrix} z\mathbb{1} - X & i\epsilon\mathbb{1} \\ i\epsilon\mathbb{1} & \bar{z}\mathbb{1} - X^\dagger \end{pmatrix}^{-1} \right\rangle \quad (30)$$

where we have introduced the block-trace operation

$$\text{Tr}_{b2} \begin{pmatrix} X & Y \\ Z & V \end{pmatrix}_{2N \times 2N} \equiv \begin{pmatrix} \text{Tr } X & \text{Tr } Y \\ \text{Tr } Z & \text{Tr } V \end{pmatrix}_{2 \times 2} \quad (31)$$

which reduces $2N \times 2N$ matrices to 2×2 ones. The elements of \mathcal{G} read explicitly:

$$\begin{aligned} \mathcal{G}_{11}(z, \bar{z}) &= \left\langle \frac{1}{N} \text{Tr} \frac{\bar{z}\mathbb{1} - X^\dagger}{(\bar{z}\mathbb{1} - X^\dagger)(z\mathbb{1} - X) + \epsilon^2\mathbb{1}} \right\rangle \\ \mathcal{G}_{1\bar{1}}(z, \bar{z}) &= \left\langle \frac{1}{N} \text{Tr} \frac{-i\epsilon\mathbb{1}}{(z\mathbb{1} - X)(\bar{z}\mathbb{1} - X^\dagger) + \epsilon^2\mathbb{1}} \right\rangle \\ \mathcal{G}_{\bar{1}1}(z, \bar{z}) &= \left\langle \frac{1}{N} \text{Tr} \frac{-i\epsilon\mathbb{1}}{(\bar{z}\mathbb{1} - X^\dagger)(z\mathbb{1} - X) + \epsilon^2\mathbb{1}} \right\rangle \\ \mathcal{G}_{\bar{1}\bar{1}}(z, \bar{z}) &= \left\langle \frac{1}{N} \text{Tr} \frac{z\mathbb{1} - X}{(z\mathbb{1} - X)(\bar{z}\mathbb{1} - X^\dagger) + \epsilon^2\mathbb{1}} \right\rangle \end{aligned} \quad (32)$$

In all these equations we tacitly assume the averages in the right hand side to be calculated in the double limit: first $N \rightarrow \infty$ and then $\epsilon \rightarrow 0$. The indices 11 , $1\bar{1}$, $\bar{1}1$ and $\bar{1}\bar{1}$ merely reflect positions of blocks in the 2×2 matrix \mathcal{G} . We see that the upper-right \mathcal{G}_{11} is equal to the Green's function $G(z, \bar{z}) = \mathcal{G}_{11}(z, \bar{z})$ (28). On the other hand, the main advantage of using the matrix \mathcal{G} is that it can be calculated using simple geometric series expansion. Indeed, defining $2N \times 2N$ matrices

$$\mathcal{Z}_\epsilon = \begin{pmatrix} z\mathbb{1} & i\epsilon\mathbb{1} \\ i\epsilon\mathbb{1} & \bar{z}\mathbb{1} \end{pmatrix} \quad (33)$$

and

$$\mathcal{H} = \begin{pmatrix} X & 0 \\ 0 & X^\dagger \end{pmatrix}. \quad (34)$$

we can see that the generalized Green's function is given formally by the same definition as the usual Green's function G but in the space of doubled dimensions

$$\mathcal{G}(z, \bar{z}) = \lim_{\epsilon \rightarrow 0} \lim_{N \rightarrow \infty} \frac{1}{N} \left\langle \text{Tr}_{b2} \frac{1}{\mathcal{Z}_\epsilon - \mathcal{H}} \right\rangle. \quad (35)$$

For the sake of the argument we have written now the double limit explicitly. As in the hermitian case, the Green's function is completely determined by the knowledge of 'generalized' moments. They are now however matrix-valued

$$\lim_{\epsilon \rightarrow 0} \lim_{N \rightarrow \infty} \frac{1}{N} \langle \text{Tr}_{b2} \mathcal{Z}_\epsilon^{-1} \mathcal{H} \mathcal{Z}_\epsilon^{-1} \mathcal{H} \dots \mathcal{Z}_\epsilon^{-1} \rangle \quad (36)$$

and are not easily related to the eigenvalue density. As before, we now proceed by applying the diagrammatic techniques to determine the non-hermitian Green's function. We begin by writing equations for generating functions for planar diagrams. In analogy to (10), we introduce the self-energy Σ but now as a matrix-valued function

$$\Sigma(z, \bar{z}) \equiv \begin{pmatrix} \Sigma_{11}(z, \bar{z}) & \Sigma_{1\bar{1}}(z, \bar{z}) \\ \Sigma_{\bar{1}1}(z, \bar{z}) & \Sigma_{\bar{1}\bar{1}}(z, \bar{z}) \end{pmatrix} \quad (37)$$

As in the hermitian case Σ is a generating function for one-line irreducible diagrams. In general it is not diagonal. Formally it is related to the Green's function as

$$\mathcal{G}(z, \bar{z}) = (\mathcal{Z} - \Sigma(z, \bar{z}))^{-1}. \quad (38)$$

where \mathcal{Z} is a diagonal 2×2 matrix

$$\mathcal{Z} = \begin{pmatrix} z & 0 \\ 0 & \bar{z} \end{pmatrix} \quad (39)$$

obtained from \mathcal{Z}_ϵ by taking block trace and setting $\epsilon = 0$. This may be done since the equation (38) is already in the limit $N \rightarrow \infty$. From here on we will set $\epsilon = 0$ in all equations.

An explicit solution for the Green's function $G(z, \bar{z}) = \mathcal{G}_{11}(z, \bar{z})$ takes therefore the following form

$$G(z, \bar{z}) = \frac{\bar{z} - \Sigma_{1\bar{1}}}{(z - \Sigma_{11})(\bar{z} - \Sigma_{\bar{1}\bar{1}}) - \Sigma_{1\bar{1}}\Sigma_{\bar{1}1}}. \quad (40)$$

We skipped arguments (z, \bar{z}) of Σ 's on the right hand side to shorten the notation. The non-diagonal terms in (30) also contain an interesting information [27], namely their product is equal to the correlator between left ($\langle L_i |$) and right ($| R_i \rangle$) eigenvectors of X , introduced originally in [28]

$$C(z, \bar{z}) \equiv -\mathcal{G}_{1\bar{1}}\mathcal{G}_{\bar{1}1} = \frac{\pi}{N} \left\langle \sum_{i=1}^N \langle L_i | L_i \rangle \langle R_i | R_i \rangle \delta^{(2)}(z - \lambda_i) \right\rangle \quad (41)$$

Since $C(z, \bar{z})$ vanishes outside the eigenvalue support, and for typical nonhermitian ensembles is nonzero, the condition $C(z, \bar{z}) = 0$ often provides a convenient equation for the boundary separating holomorphic and nonholomorphic solutions of the spectral problem. Indeed, when off-diagonal terms of Σ vanish equation (40) simplifies to that for hermitian matrices $G = 1/(z - \Sigma_{11})$.

As in the hermitian case we can write an independent equation relating \mathcal{G} and Σ – a counterpart of (11). The \mathcal{R} transform however is now a more complicated object since it maps a 2×2 matrix \mathcal{G} onto a 2×2 matrix Σ :

$$\Sigma(z, \bar{z}) = \mathcal{R}(\mathcal{G}(z, \bar{z})) \quad (42)$$

or in an explicit notation

$$\begin{pmatrix} \Sigma_{11}(z, \bar{z}) & \Sigma_{1\bar{1}}(z, \bar{z}) \\ \Sigma_{\bar{1}1}(z, \bar{z}) & \Sigma_{\bar{1}\bar{1}}(z, \bar{z}) \end{pmatrix} = \begin{pmatrix} \mathcal{R}_{11}(\mathcal{G}(z, \bar{z})) & \mathcal{R}_{1\bar{1}}(\mathcal{G}(z, \bar{z})) \\ \mathcal{R}_{\bar{1}1}(\mathcal{G}(z, \bar{z})) & \mathcal{R}_{\bar{1}\bar{1}}(\mathcal{G}(z, \bar{z})) \end{pmatrix} \quad (43)$$

In order to complete the analogy to the hermitian case we shall now provide a diagrammatic interpretation of the last relation.

B. Planar Feynman diagrams for non-hermitian matrices

We shall now discuss the diagrammatic method of calculating eigenvalue densities for non-hermitian random matrices generated by probability measures of the type $P(X) \sim \exp(-N\text{Tr}V(X, X^\dagger))$ in the limit $N \rightarrow \infty$, which as before corresponds to the limit of planar diagrams. We consider potentials given by sums of terms being alternating sequences of powers of X and X^\dagger like $XX^\dagger X^2 X^\dagger \dots$. Such a potential must be hermitian $[V(X, X^\dagger)]^\dagger = V(X, X^\dagger)$ to ensure that the expression in the exponent is a real number. The first step of the diagrammatic construction is to split the measure into the Gaussian part and the residual one $P(X) = P_0(X)P_r(X)$ and use $P_0(X)$ to calculate averages $\langle \dots \rangle_0$ which can be represented as Feynman diagrams, exactly as for hermitian matrices (15). The Gaussian measure $P_0(V) \sim e^{-N\text{Tr}V_0(X)}$ is constructed from a quadratic potential. The most general form of a quadratic potential being a real number is $\text{Tr}V_0(X) = a\text{Tr}XX^\dagger + b\text{Tr}(X^2 + X^{\dagger 2})$ with some real coefficients a, b . The coefficients must be appropriately chosen to ensure the potential be positive. The expression is manifestly positive when expressed in new parameters $\sigma, \tau \in (-1, 1)$:

$$P_0(X) \sim \exp \left\{ -N \frac{1}{\sigma^2} \frac{1}{1 - \tau^2} \left(\text{Tr}XX^\dagger - \tau \frac{1}{2} \text{Tr}(XX + X^\dagger X^\dagger) \right) \right\} \quad (44)$$

as one can see for example by writing down the corresponding two-point correlation functions (propagators):

$$\begin{aligned} \langle X_{ab}X_{cd}^\dagger \rangle_0 &= \langle X_{ab}^\dagger X_{cd} \rangle_0 = \frac{\sigma^2}{N} \delta_{ad} \delta_{bc}, \\ \langle X_{ab}X_{cd} \rangle_0 &= \langle X_{ab}^\dagger X_{cd}^\dagger \rangle_0 = \tau \cdot \frac{\sigma^2}{N} \delta_{ad} \delta_{bc} \end{aligned} \quad (45)$$

The propagators represent elementary building blocks of Feynman diagrams. As for hermitian matrices the propagators are proportional to delta functions, so after taking the block trace we can reduce the problem to a 2×2 one with propagators corresponding to $XX^\dagger, X^\dagger X, XX, X^\dagger X^\dagger$. The crucial step in inferring the index structure of equations relating 2×2 matrices \mathcal{G} and Σ is to use the correspondence between $X \leftrightarrow \mathcal{H}_{11}$ and $X^\dagger \leftrightarrow \mathcal{H}_{\bar{1}\bar{1}}$ which follows from equation (34). Let us do that. The two-point functions (45) reduce to propagators represented by double arcs shown

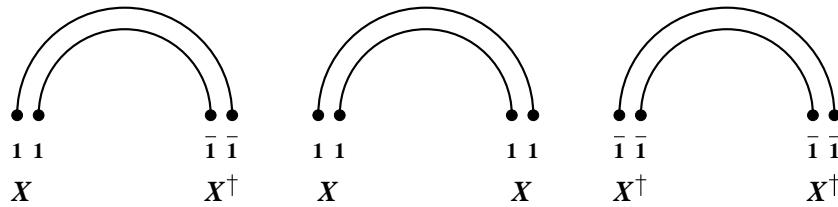


FIG. 6. Propagators for non-hermitian models generated by the Gaussian part of the measure (44). They are obtained by identification $X \leftrightarrow \mathcal{H}_{11}$ and $X^\dagger \leftrightarrow \mathcal{H}_{\bar{1}\bar{1}}$ (34). This identification induces the indexing of the endpoints marked as dots in the figure (Left) $\langle XX^\dagger\rangle_0 = \langle \mathcal{H}_{11}\mathcal{H}_{\bar{1}\bar{1}}\rangle_0 = \sigma^2$; (Middle) $\langle XX\rangle_0 = \langle \mathcal{H}_{11}\mathcal{H}_{11}\rangle_0 = \tau\sigma^2$; (Right) $\langle X^\dagger X^\dagger\rangle_0 = \langle \mathcal{H}_{\bar{1}\bar{1}}\mathcal{H}_{\bar{1}\bar{1}}\rangle_0 = \tau\sigma^2$.

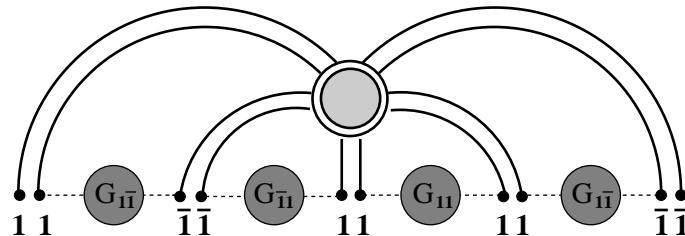


FIG. 7. Connected diagrams generated by the fifth order planar cumulant $\langle\langle XX^\dagger XXX^\dagger\rangle\rangle$ (the head of the spider). These diagrams contribute a factor $\mathcal{G}_{1̄1}\mathcal{G}_{̄1̄1}\mathcal{G}_{11}\mathcal{G}_{̄11}$ to $\Sigma_{1̄1} = \mathcal{R}_{1̄1}(\mathcal{G})$.

in figure 6. The matrix \mathcal{Z}^{-1} (39) generates lines between 11 vertices which contribute $1/z$ and lines between $\bar{1}\bar{1}$ which contribute $1/\bar{z}$, while there are no lines between mixed vertices. Using these elementary blocks we can draw graphical equations as those in figures 2 and 4. The only difference as compared to the hermitian case is that they are written for 2×2 matrices. Each black dot in the diagrams in these figures is ascribed to an index which may assume two values: either 1 or $\bar{1}$. Each pair of neighboring dots on the horizontal line in figure 4 corresponds to X or X^\dagger or to \mathcal{H}_{11} or $\mathcal{H}_{\bar{1}\bar{1}}$ as follows from the assignment (34). As an example consider a spider diagram of order five generated in the expansion shown in figure 4. Each leg of the spider may be attached to X or X^\dagger , so on the horizontal line we have a sequence of these symbols – for instance $XX^\dagger XXX^\dagger$, or equivalently $\mathcal{H}_{11}\mathcal{H}_{\bar{1}\bar{1}}\mathcal{H}_{11}\mathcal{H}_{11}\mathcal{H}_{\bar{1}\bar{1}}$ (34). The corresponding diagram is shown in figure 7. In a shorthand notation the diagram is determined by a sequence of pairs 11, $\bar{1}\bar{1}$, 11, 11, 11 on the horizontal line which begins with 1 and ends with $\bar{1}$ so it contributes to $\Sigma_{1̄1}$, since the corresponding diagram is one-line irreducible. As one can see from the figure its contribution is proportional to $\mathcal{G}_{1̄1}\mathcal{G}_{̄1̄1}\mathcal{G}_{11}\mathcal{G}_{̄11}$. The indices of \mathcal{G} bubbles are enforced by indices of the spider legs – they must match the sequence on the horizontal line.

All such contributions are captured by a matrix valued function $\mathcal{R}(\mathcal{G})$, in this particular case by its element $\mathcal{R}_{1̄1}(\mathcal{G})$ which contains contributions generated by sequences beginning with 1 and ending with $\bar{1}$. Each element of the matrix $\mathcal{R}(\mathcal{G})$ may depend on all elements of the matrix \mathcal{G} so this function maps 2×2 matrices onto 2×2 matrices and in general is highly nontrivial (42). The exception is the Gaussian case for which the map is linear.

For the purpose of this paper let us study Gaussian case in more detail. The most general Gaussian ensemble (44) leads through (45) to (see figure 6)

$$\mathcal{R}(\mathcal{G}) = \begin{pmatrix} \Sigma_{11} & \Sigma_{1̄1} \\ \Sigma_{̄11} & \Sigma_{̄1̄1} \end{pmatrix} = \begin{pmatrix} \tau\sigma^2\mathcal{G}_{11} & \sigma^2\mathcal{G}_{1̄1} \\ \sigma^2\mathcal{G}_{̄11} & \tau\sigma^2\mathcal{G}_{̄1̄1} \end{pmatrix} \quad (46)$$

Let us now constrain ourselves to the so-called Ginibre-Girko ensemble which corresponds to the case $\tau = 0$ and $\sigma = 1$ in (6), so the matrix Σ reads

$$\mathcal{R}(\mathcal{G}) = \begin{pmatrix} \Sigma_{11} & \Sigma_{1̄1} \\ \Sigma_{̄11} & \Sigma_{̄1̄1} \end{pmatrix} = \begin{pmatrix} 0 & \mathcal{G}_{1̄1} \\ \mathcal{G}_{̄11} & 0 \end{pmatrix} \quad (47)$$

where the off-diagonal contributions are analogous to the relation $R(G) = G$ for the hermitian Gaussian ensemble. Solving (38-47) determines the spectral problem for the Ginibre-Girko ensemble. Inserting (47) into (38) we get:

$$\begin{pmatrix} \mathcal{G}_{11} & \mathcal{G}_{1̄1} \\ \mathcal{G}_{̄11} & \mathcal{G}_{̄1̄1} \end{pmatrix} = \frac{1}{|z|^2 - \mathcal{G}_{1̄1}\mathcal{G}_{̄1̄1}} \cdot \begin{pmatrix} \bar{z} & \mathcal{G}_{1̄1} \\ \mathcal{G}_{̄11} & z \end{pmatrix} \quad (48)$$

The equation for off-diagonal element reads

$$\mathcal{G}_{1\bar{1}} = \frac{\mathcal{G}_{1\bar{1}}}{|z|^2 - \mathcal{G}_{1\bar{1}}\mathcal{G}_{\bar{1}\bar{1}}} . \quad (49)$$

It has two-solutions: one with $\mathcal{G}_{1\bar{1}} = 0$ and the another one with $\mathcal{G}_{1\bar{1}} \neq 0$. The first one leads to a holomorphic Green's function $G = \mathcal{G}_{11}$

$$G(z) = \frac{1}{z} \quad (50)$$

while the second one to a non-holomorphic (see the upper diagonal component of equation $G \equiv \mathcal{G}_{11}$ (48))

$$G(z, \bar{z}) = \bar{z} \quad (51)$$

which gives the following eigenvalue density

$$\rho(x, y) = \frac{1}{\pi} \frac{\partial}{\partial \bar{z}} \mathcal{G}_{11}(z, \bar{z}) = \frac{1}{\pi} . \quad (52)$$

Both solutions match at the boundary $|z|^2 = 1$. So we have recovered a known result that the complex eigenvalues of the Ginibre-Girko ensemble are uniformly distributed on the unit disc.

C. Addition law for non-hermitian matrices

One can actually use exactly the same arguments as for hermitian matrices to deduce the law of free addition for non-hermitian matrices. It has a simple form given in terms of matrix-valued R transforms:

$$\mathcal{R}_{A+B}(\mathcal{G}) = \mathcal{R}_A(\mathcal{G}) + \mathcal{R}_B(\mathcal{G}) \quad (53)$$

which follows from the fact that all mixed AB propagators vanish and therefore all mixed connected diagrams having a line between A and B vanish too. Since such diagrams represent connected moments (free cumulants), e.g. $\frac{1}{N} \langle \langle AB^2 A^\dagger AB^\dagger \rangle \rangle = 0$, we see that the only non-zero contributions come from connected diagrams (moments) which either have all A 's or all B 's. For applications and more details of this generalized addition law we refer to [12–14].

V. MULTIPLICATION LAW

A. Preliminaries

The S transform plays the same role for matrix multiplication as the R transform for addition. Assume that A and B are large independent (free) random matrices given by a product measure (18). The multiplication law tells us how to calculate spectral moments $\frac{1}{N} \langle \text{Tr}(AB)^n \rangle$ of the product $H = AB$ provided we know the spectral moments of A and B or equivalently that we know the corresponding Green's functions $G_A(z)$ and $G_B(z)$. The multiplication law, expressed in terms of the S transform, reads [1]

$$S_{A \cdot B}(z) = S_A(z)S_B(z) \quad (54)$$

and the S transform is defined by

$$S(z) = \frac{1+z}{z} \chi(z), \quad \text{where} \quad \chi(zG(z) - 1) = \frac{1}{z} . \quad (55)$$

The algorithm for "multiplication" is similar to that for "addition":

- (i) Calculate $S_A(z)$ and $S_B(z)$ using (55).
- (ii) Use the multiplication law (54).
- (iii) Use again (55) to derive $G_{AB}(z)$ for the product of AB .

Let us first derive some useful relations between the R and S transforms. Changing variables $z = yG(y) - 1$ in (55) we get

$$S(yG(y) - 1) = \frac{1}{y - \frac{1}{G(y)}} . \quad (56)$$

Using (10) we can rewrite the last equation as

$$S(G(y)\Sigma(y)) = \frac{1}{\Sigma(y)} . \quad (57)$$

Setting $\Sigma(z) = R(G(z))$ and taking the reciprocals of both sides we arrive at

$$\frac{1}{S(G(y)R(G(y)))} = R(G(y)) . \quad (58)$$

Changing variables once again to $z = G(y)$ we obtain the equation

$$R(z) = \frac{1}{S(zR(z))} \quad (59)$$

which gives an explicit relation between the R and S transforms. The S transform can be defined only if the R transform does not vanish at the origin: $R(0) \neq 0$. This corresponds to random matrices with a non-vanishing first moment (cumulant) $\frac{1}{N}\langle \text{Tr}H \rangle = \frac{1}{N}\langle \langle \text{Tr}H \rangle \rangle \neq 0$. Otherwise the S transform cannot be defined as a power series and all the manipulations presented above break down. The last equation can be inverted. Let us introduce a new variable $y = zR(z)$. Now (59) reads

$$S(y) = \frac{1}{R\left(\frac{y}{R(z)}\right)} = \frac{1}{R\left(\frac{y}{R\left(\frac{y}{R(z)}\right)}\right)} = \frac{1}{R\left(\frac{y}{R\left(\frac{y}{R\left(\frac{y}{R(z)}\right)}\right)}\right)} \quad (60)$$

where z can be recursively eliminated by repeating the substitution $z = \frac{y}{R(z)}$ ad infinitum. This leads to a function which is nested infinitely many times forming a sort of continued fraction. The last equation can alternatively be written as

$$S(z) = \frac{1}{R(zS(z))} \quad (61)$$

which is an inverse formula to (59). The two equations can be written in a symmetric way as mutually inverse maps

$$z = yS(y) \quad \text{and} \quad y = zR(z) . \quad (62)$$

As an example we consider a shifted Gaussian random matrix which has only two first non-vanishing cumulants. For the standardized choice $\kappa_1 = \kappa_2 = 1$ the R transform reads $R(z) = 1 + z$. Using (61) we obtain

$$S(z) = \frac{1}{1 + zS(z)} \quad (63)$$

and hence $S(z) = \frac{-1 + \sqrt{1 + 4z}}{2z}$.

B. Diagrammatic derivation of the multiplication law

We are now ready to diagrammatically derive the S transform and the corresponding multiplication law. The argument given below will turn out to be crucial for the generalization to non-hermitian matrices. The initial point of the construction is to consider a $2N \times 2N$ block matrix \mathcal{H} and its even powers¹

$$\mathcal{H} = \begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix} , \quad \mathcal{H}^{2k} = \begin{pmatrix} (AB)^k & 0 \\ 0 & (BA)^k \end{pmatrix} . \quad (64)$$

The upper-left corner of \mathcal{H}^{2k} involves solely the powers of AB , which we are interested in. In order to have an access to the traces of individual blocks in the matrix we again apply the block trace operation defined before. The upper-left corner of the reduced matrix $\text{Tr}_{b2}\mathcal{H}^{2k}$ is equal $\text{Tr}(AB)^k$ while of $\text{Tr}_{b2}\mathcal{H}^{2k+1}$ is equal zero. So now the idea is to reformulate the problem of calculating the Green's function for the product

$$G_{AB}(z) = \frac{1}{N} \left\langle \text{Tr} \frac{1}{z\mathbb{1} - AB} \right\rangle \quad (65)$$

¹ This should not be confused with the $2N \times 2N$ block matrix constructed for the nonhermitian Random Matrix Ensembles in section IV.

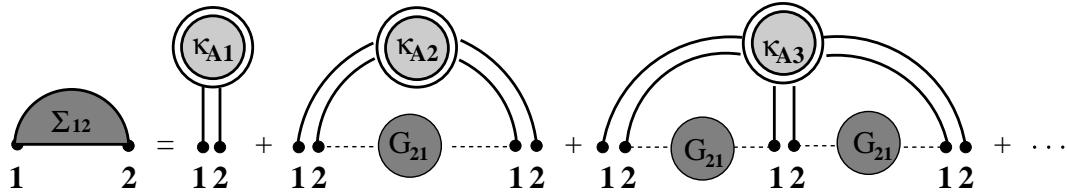


FIG. 8. The spider diagrams correspond to free cumulants generated by the matrix $A = \mathcal{H}_{12}$ and therefore the double dots on the horizontal line are indexed by 12. So on the horizontal line we have alternating indices 1212...12 and this enforces all the \mathcal{G} -bubbles to have indices 21 as one can see in the figure. Therefore there are only \mathcal{G}_{21} bubbles in the diagram and the corresponding equation is $\Sigma_{12}(w) = R_A(\mathcal{G}_{21})$. The analogous equation for B -cumulants is $\Sigma_{21}(w) = R_B(\mathcal{G}_{12})$. Similarly, one can see that $\Sigma_{11}(w) = \Sigma_{22}(w) = 0$ since one of the double dots on the horizontal line would need to have identical indices, for which as we know from (64) the double line is equal zero.

as a problem of calculating the upper-left corner of the Green's function $\mathcal{G}(w)$ for the matrix \mathcal{H} :

$$\mathcal{G}(w) = \begin{pmatrix} \mathcal{G}_{11}(w) & \mathcal{G}_{12}(w) \\ \mathcal{G}_{21}(w) & \mathcal{G}_{22}(w) \end{pmatrix} = \frac{1}{N} \left\langle \text{Tr}_{b2} \frac{1}{w \mathbb{1} - \mathcal{H}} \right\rangle. \quad (66)$$

where w is a complex number and $\mathbb{1}$ is a unity matrix of dimensions $2N \times 2N$. One can easily check that

$$G_{AB}(z = w^2) = \frac{\mathcal{G}_{11}(w)}{w} \quad (67)$$

since only every second (even) power of $\mathcal{G}(w)$ contributes to the power expansion of $\mathcal{G}_{11}(w)$, which is thus a power expansion in $z = w^2$.

The next step is to define self-energy $\Sigma(w)$. It is a 2×2 matrix

$$\Sigma(w) \equiv \begin{pmatrix} \Sigma_{11}(w) & \Sigma_{12}(w) \\ \Sigma_{21}(w) & \Sigma_{22}(w) \end{pmatrix} \quad (68)$$

that is related to the Green's function as

$$\mathcal{G}(w) = (w \mathbb{1} - \Sigma(w))^{-1} \quad (69)$$

in analogy to (10). All the matrices in the last equation are of 2×2 dimensions. This is the first Dyson-Schwinger equation. To write down the second one – a counterpart of (11) it is convenient to use its diagrammatic representation as that in figure 4. Instead of a scalar equation (11) we will have a matrix equation for 2×2 matrices \mathcal{G} and Σ (66) and (68). Since we have now have 2×2 matrices it is crucial to work out the index structure of the corresponding equation. This structure stems from the correspondence $A \leftrightarrow \mathcal{H}_{12}$ and $B \leftrightarrow \mathcal{H}_{21}$ that follows from the position of the blocks in \mathcal{H} (64). Note the difference to the case discussed in the previous section were we had diagonal blocks (34).

The only non-vanishing cumulants are $\frac{1}{N} \langle \langle \text{Tr} \mathcal{H}_{12}^n \rangle \rangle = \frac{1}{N} \langle \langle \text{Tr} A^n \rangle \rangle \equiv \kappa_{A,n}$ or $\frac{1}{N} \langle \langle \text{Tr} \mathcal{H}_{21}^n \rangle \rangle = \frac{1}{N} \langle \langle \text{Tr} B^n \rangle \rangle \equiv \kappa_{B,n}$ while all mixed ones vanish as we discussed in the previous section. Due to this, the index structure of non-vanishing one-line-irreducible diagrams is restricted to that shown in figure 8 and its counterpart obtained by exchanging $1 \leftrightarrow 2$ and $A \leftrightarrow B$. The diagrammatic equations discussed in figure 8 can be summarized as

$$\Sigma(w) = \begin{pmatrix} 0 & R_A(\mathcal{G}_{21}(w)) \\ R_B(\mathcal{G}_{12}(w)) & 0 \end{pmatrix}. \quad (70)$$

Inserting this into (69) yields

$$\begin{pmatrix} \mathcal{G}_{11}(w) & \mathcal{G}_{12}(w) \\ \mathcal{G}_{21}(w) & \mathcal{G}_{22}(w) \end{pmatrix} = \begin{pmatrix} w & -R_A(\mathcal{G}_{21}(w)) \\ -R_B(\mathcal{G}_{12}(w)) & w \end{pmatrix}^{-1} \quad (71)$$

which gives a direct relation between the Green's function $\mathcal{G}(w)$ and the R transform.

We shall now rewrite this equation in a way which explicitly exhibits multiplicative structure. Note that in the following manipulations we do not need to assume anything about the first moment i.e. whether the ensemble is centered or not. Inverting the matrix on the right hand side we obtain

$$\mathcal{G}_{12}(w) = \frac{1}{\text{Det}} R_A(\mathcal{G}_{21}(w)), \quad \mathcal{G}_{21}(w) = \frac{1}{\text{Det}} R_B(\mathcal{G}_{12}(w)) \quad (72)$$

$$\mathcal{G}_{11}(w) = \mathcal{G}_{22}(w) = \frac{w}{\text{Det}} . \quad (73)$$

where Det is the determinant of the matrix, $w\mathbb{1} - \Sigma(w)$, on the right hand side of (71):

$$\text{Det} = w^2 - R_A(\mathcal{G}_{21}(w)) R_B(\mathcal{G}_{12}(w)) . \quad (74)$$

Inserting two last equations to (67) we obtain

$$G_{AB}(z) = \frac{\mathcal{G}_{11}(w)}{w} = \frac{1}{\text{Det}} = \frac{1}{z - R_A(\mathcal{G}_{21}(w)) R_B(\mathcal{G}_{12}(w))} \quad (75)$$

where $z = w^2$. Comparing the denominator in this equation to that of the standard equation $G_{AB}(z) = 1/(z - R_{AB}(G_{AB}(z)))$ (13) we get

$$R_{AB}(G_{AB}(z)) = R_A(\mathcal{G}_{21}(w)) R_B(\mathcal{G}_{12}(w)) . \quad (76)$$

At this stage we see the first hint of a multiplicative structure emergence. In order to complete this equation we also need (72). Let us set $g = G_{AB}(z)$, $g_A = \mathcal{G}_{12}(w)$ and $g_B = \mathcal{G}_{21}(w)$ to simplify arguments in the R transforms in the last equation. Using this substitution we can write (76) and (72) in a compact form as a closed set of equations for the R transform of the product

$$R_{AB}(g) = R_A(g_B) R_B(g_A) \quad (77)$$

and

$$g_A = g R_A(g_B) , \quad g_B = g R_B(g_A) . \quad (78)$$

which is equivalent to (3) announced at the beginning of the paper. This is the multiplication law formulated in terms of the R transform. Its main advantage in comparison to the S transform is that it can be applied even to centered ensembles (i.e. having vanishing mean) including the case when both are centered (see the examples in sections VIA and VIB).

The difference with respect to the conventional multiplication law $S_{AB}(z) = S_A(z) S_B(z)$ is that the individual factors appearing in (77) are not expressed uniquely in terms of the properties of a *single* random matrix ensemble e.g. the factor $R_A(\cdot)$ is evaluated on g_B which is related to the ensemble B . However it is straightforward to obtain from (77)-(78) the conventional multiplication law as we shall illustrate below.

Let us introduce a new variable $y = g R_{AB}(g)$. We can now express g_B – the argument of R_A purely in terms of the properties of ensemble A :

$$g_B = g R_B(g_A) = g \frac{R_{AB}(g)}{R_A(g_B)} = \frac{y}{R_A(g_B)} = \frac{y}{R_A\left(\frac{y}{R_A(\dots)}\right)} \quad (79)$$

Now each of the factors in (77) depends on a single ensemble. We may do the same for the left hand side, which becomes (60)

$$R_{AB}\left(\frac{y}{R_{AB}(g)}\right) = R_{AB}\left(\frac{y}{R_{AB}\left(\frac{y}{R_{AB}(\dots)}\right)}\right) = \frac{1}{S_{AB}(y)} \quad (80)$$

Putting these formulas together, we can finally write (77) using only the variable y [31]

$$R_{AB}\left(\frac{y}{R_{AB}\left(\frac{y}{R_{AB}(\dots)}\right)}\right) = R_A\left(\frac{y}{R_A\left(\frac{y}{R_A(\dots)}\right)}\right) R_B\left(\frac{y}{R_B\left(\frac{y}{R_B(\dots)}\right)}\right) \quad (81)$$

which amounts to the standard formulation for the multiplication law [1] as follows from (60)

$$S_{AB}(y) = S_A(y) S_B(y) . \quad (82)$$

The necessity of assuming noncentered distributions comes from the fact that the implicit continued fractions appearing in (60) make sense only for $R(z) \sim \text{const} + \mathcal{O}(z)$ with nonzero constant term [29].

C. Multiplication law for non-hermitian matrices

In order to derive the multiplication law for non-hermitian matrices we combine the two formalisms outlined in previous sections. First we define a $2N \times 2N$ matrix \mathcal{D} in analogy to (64)

$$\mathcal{D} = \begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix}_{2N \times 2N} \quad (83)$$

and then duplicate it using (34) to obtain an extended Green's function for non-hermitian matrices. This technique has been introduced in [21] and used for specific ensembles [19, 21]. In this paper we will use it to obtain a multiplication law for arbitrary (free) nonhermitian matrices².

This procedure leads to a four-fold matricial structure ("double doubling") where the primary object is a $4N \times 4N$ matrix

$$\mathcal{H} = \begin{pmatrix} \mathcal{D} & 0 \\ 0 & \mathcal{D}^\dagger \end{pmatrix} = \begin{pmatrix} 0 & A & 0 & 0 \\ B & 0 & 0 & 0 \\ 0 & 0 & 0 & B^\dagger \\ 0 & 0 & A^\dagger & 0 \end{pmatrix}_{4N \times 4N} \quad (84)$$

and the corresponding Green's function

$$\mathcal{G}(w, \bar{w}) = \left\langle \left[\begin{pmatrix} w\mathbb{1} & 0 & 0 & 0 \\ 0 & w\mathbb{1} & 0 & 0 \\ 0 & 0 & \bar{w}\mathbb{1} & 0 \\ 0 & 0 & 0 & \bar{w}\mathbb{1} \end{pmatrix} - \begin{pmatrix} 0 & A & 0 & 0 \\ B & 0 & 0 & 0 \\ 0 & 0 & 0 & B^\dagger \\ 0 & 0 & A^\dagger & 0 \end{pmatrix} \right]^{-1} \right\rangle \quad (85)$$

Using the block-trace operation tr_{b4} we reduce the problem to calculations for 4×4 matrices

$$\mathcal{G}(\mathcal{W}) \equiv \begin{pmatrix} \mathcal{G}_{11} & \mathcal{G}_{12} & \mathcal{G}_{1\bar{1}} & \mathcal{G}_{1\bar{2}} \\ \mathcal{G}_{21} & \mathcal{G}_{22} & \mathcal{G}_{2\bar{1}} & \mathcal{G}_{2\bar{2}} \\ \mathcal{G}_{\bar{1}1} & \mathcal{G}_{\bar{1}2} & \mathcal{G}_{\bar{1}\bar{1}} & \mathcal{G}_{\bar{1}\bar{2}} \\ \mathcal{G}_{\bar{2}1} & \mathcal{G}_{\bar{2}2} & \mathcal{G}_{\bar{2}\bar{1}} & \mathcal{G}_{\bar{2}\bar{2}} \end{pmatrix}_{4 \times 4} = \frac{1}{N} \text{tr}_{b4} \mathcal{G}(w, \bar{w}) \quad (86)$$

where $\mathcal{W} = \text{diag}(w, w, \bar{w}, \bar{w})$. The labeling of the matrix elements follows the convention adopted in the previous sections. Similarly, we define a self-energy $\Sigma(\mathcal{W})$ as a 4 by 4 matrix:

$$\mathcal{G}(W) = (\mathcal{W} - \Sigma(\mathcal{W}))^{-1} \quad (87)$$

which is related to a 4 by 4 matrix representing the generalized R transform:

$$\Sigma(\mathcal{W}) = \mathcal{R}(\mathcal{G}(\mathcal{W})). \quad (88)$$

The elements of Σ and \mathcal{R} are indexed in the same way as the elements of \mathcal{G} (86).

We exploit these 4×4 matrices as auxiliary objects to derive relations between 2×2 Green's functions $\mathcal{G}_A(\mathcal{Z})$, $\mathcal{G}_B(\mathcal{Z})$ and $\mathcal{G}_M(\mathcal{Z})$ for A , B and the product $M = AB$. The naming convention for elements of 2×2 matrices

$$\mathcal{G}_A(\mathcal{Z}) = \begin{pmatrix} \mathcal{G}_{(A)11} & \mathcal{G}_{(A)1\bar{1}} \\ \mathcal{G}_{(A)\bar{1}1} & \mathcal{G}_{(A)\bar{1}\bar{1}} \end{pmatrix} \quad (89)$$

is a bit inconvenient since it requires three subscripts for each element. To avoid multiple subscripts like $(A)1\bar{1}$ we introduce a shorthand notation substituting multiples indices like $(A)1\bar{1}$ by $A\bar{A}$ etc. In this new notation a double subscript identifies both the matrix for which the generating function is calculated and the position of the element. Using this convention we have

$$\mathcal{G}_A(\mathcal{Z}) = \begin{pmatrix} \mathcal{G}_{AA} & \mathcal{G}_{A\bar{A}} \\ \mathcal{G}_{\bar{A}A} & \mathcal{G}_{\bar{A}\bar{A}} \end{pmatrix} \quad (90)$$

² To remind the reader, 'free' means essentially that the probability distributions of the two ensembles are independent *and* that we take the $N \rightarrow \infty$ limit.

and similarly for two remaining generating functions

$$\Sigma_A(\mathcal{Z}) = \begin{pmatrix} \Sigma_{AA} & \Sigma_{A\bar{A}} \\ \Sigma_{\bar{A}A} & \Sigma_{\bar{A}\bar{A}} \end{pmatrix}, \quad \mathcal{R}_A(\mathcal{G}) = \begin{pmatrix} \mathcal{R}_{AA} & \mathcal{R}_{A\bar{A}} \\ \mathcal{R}_{\bar{A}A} & \mathcal{R}_{\bar{A}\bar{A}} \end{pmatrix}. \quad (91)$$

We use the same convention for all matrices, including B and M . For brevity we skipped the arguments of the matrix elements on the right hand side of the equations above. We tacitly assumed that they are identical as on the left hand side. We will frequently use this shorthand notation below.

To summarize the notation, \mathcal{R}_M denotes a 2×2 matrix of the R transform for M while \mathcal{R}_{MM} – its upper left element, etc. For 4×4 matrices like $\mathcal{G}(\mathcal{W})$, $\Sigma(\mathcal{W})$ and $\mathcal{R}(\mathcal{G})$ we instead use the indexing as in (86) which uniquely identifies the positions of elements in such 4×4 matrices. The link between the two conventions emerges from the equation (84)

$$\mathcal{H} \equiv \begin{pmatrix} \mathcal{H}_{11} & \mathcal{H}_{12} & \mathcal{H}_{1\bar{1}} & \mathcal{H}_{1\bar{2}} \\ \mathcal{H}_{21} & \mathcal{H}_{22} & \mathcal{H}_{2\bar{1}} & \mathcal{H}_{2\bar{2}} \\ \mathcal{H}_{\bar{1}1} & \mathcal{H}_{\bar{1}2} & \mathcal{H}_{\bar{1}\bar{1}} & \mathcal{H}_{\bar{1}\bar{2}} \\ \mathcal{H}_{\bar{2}1} & \mathcal{H}_{\bar{2}2} & \mathcal{H}_{\bar{2}\bar{1}} & \mathcal{H}_{\bar{2}\bar{2}} \end{pmatrix} = \begin{pmatrix} 0 & A & 0 & 0 \\ B & 0 & 0 & 0 \\ 0 & 0 & 0 & B^\dagger \\ 0 & 0 & A^\dagger & 0 \end{pmatrix} \quad (92)$$

that allows us to identify $A \leftrightarrow \mathcal{H}_{12}$, $A^\dagger \leftrightarrow \mathcal{H}_{\bar{2}\bar{1}}$ and $B \leftrightarrow \mathcal{H}_{21}$, $B^\dagger \leftrightarrow \mathcal{H}_{\bar{1}\bar{2}}$. We use this identification to rewrite the equation (88) in terms of 2×2 matrices. We begin by noting that even powers \mathcal{H}^{2k} of \mathcal{H} (92) generate powers M^k of the product $M = AB$ in the upper left corner of the block matrices

$$\mathcal{H}^{2k} = \begin{pmatrix} (AB)^k & 0 & 0 & 0 \\ 0 & (BA)^k & 0 & 0 \\ 0 & 0 & (AB)^\dagger k & 0 \\ 0 & 0 & 0 & (BA)^\dagger k \end{pmatrix} \quad (93)$$

These moments are generated by the element $\mathcal{G}_{11}(\mathcal{W})$ of the 4×4 Green's function $\mathcal{G}(\mathcal{W})$ (86) or alternatively by the element $\mathcal{G}_{MM}(\mathcal{Z})$ of the 2×2 Green's function $\mathcal{G}_M(\mathcal{Z})$, so we have

$$G_{AB}(z, \bar{z}) = \mathcal{G}_{MM}(\mathcal{Z}) = \frac{\mathcal{G}_{11}(\mathcal{W})}{w} \quad (94)$$

where $\mathcal{Z} = \text{diag}(z, \bar{z})$, $\mathcal{W} = \text{diag}(w, w, \bar{w}, \bar{w})$ and $z = w^2$, analogously to (67). This equation, allows us to determine Green's function $G_{AB}(z, \bar{z})$ and additionally provides a link between \mathcal{G}_M and \mathcal{G}_A and \mathcal{G}_B since elements of the 4×4 Green's function \mathcal{G} can be explicitly expressed in terms of \mathcal{G}_A and \mathcal{G}_B , as we will see below using planar Feynman diagrams.

First we recall that all mixed connected diagrams vanish since AB propagators are equal zero. The last statement means that there are no direct lines in the diagram connecting A and B vertices. All non-vanishing connected diagrams are either of A -type like $\langle\langle \frac{1}{N} \text{tr} AA^\dagger AA \dots \rangle\rangle$ or B -type like $\langle\langle \frac{1}{N} \text{tr} BB^\dagger BB \dots \rangle\rangle$. They are generated by alternating sequences either of A and A^\dagger or of B and B^\dagger but not mixed ones. In other words there are only A -spider or B -spider diagrams. In the \mathcal{H} notation the first type is generated by sequences of \mathcal{H}_{12} and $\mathcal{H}_{\bar{2}\bar{1}}$ while the second type of \mathcal{H}_{21} and $\mathcal{H}_{\bar{1}\bar{2}}$ as follows from the correspondance (92). We show in figure 9 an example of a diagram contributing to the left hand side of equation (88). More generally, diagrams with a A -spider have on the horizontal line alternating sequences like $\mathcal{H}_{12}\mathcal{H}_{\bar{2}\bar{1}}\mathcal{H}_{\bar{2}\bar{1}}\mathcal{H}_{12}\mathcal{H}_{12} \dots$ which are sandwiched by $\mathcal{G}_{2\bar{2}}$, $\mathcal{G}_{\bar{1}\bar{2}}$, $\mathcal{G}_{\bar{1}1}$, \mathcal{G}_{21}, \dots which match the index sequence. The left most index in the sequence of \mathcal{H} 's may be equal 1 or $\bar{2}$ and the right most 2 or $\bar{1}$ so the corresponding diagrams contribute to Σ_{12} , $\Sigma_{1\bar{1}}$, $\Sigma_{\bar{2}2}$ or $\Sigma_{\bar{2}\bar{1}}$. Diagrams with a B -spider have sequences like $\mathcal{H}_{21}\mathcal{H}_{\bar{1}\bar{2}} \dots \mathcal{H}_{21}$ etc, whose left most index is either 2 or $\bar{1}$ and the right most index is either 1 or $\bar{2}$, so the corresponding diagrams contribute to Σ_{21} , $\Sigma_{\bar{2}\bar{2}}$, $\Sigma_{\bar{1}1}$ or $\Sigma_{\bar{1}\bar{2}}$. All others Σ 's must be equal zero

$$\begin{aligned} \Sigma_{11} &= \Sigma_{22} = \Sigma_{\bar{1}\bar{1}} = \Sigma_{\bar{2}\bar{2}} = 0 \\ \Sigma_{1\bar{2}} &= \Sigma_{\bar{2}\bar{1}} = \Sigma_{\bar{1}2} = \Sigma_{\bar{2}1} = 0 \end{aligned} \quad (95)$$

since there are no mixed AB -spiders. Coming back to the equations for Σ_{12} , $\Sigma_{1\bar{1}}$, $\Sigma_{\bar{2}2}$, $\Sigma_{\bar{2}\bar{1}}$ generated by the A -spider we notice that the indices of the \mathcal{G} bubbles which enter the sandwich between the spider legs have complementary indices $\mathcal{G}_{2\bar{2}}$, $\mathcal{G}_{\bar{1}\bar{2}}$, $\mathcal{G}_{\bar{1}1}$, \mathcal{G}_{21} as compared to those of Σ 's. The same holds for equations for indices of \mathcal{G} 's and Σ 's generated by the B -spider. Moreover, if we compare indices of Σ 's for A spiders to \mathcal{G} 's for B spiders we see they are identical, and the same holds for Σ 's for B spiders and \mathcal{G} 's for A spiders. All these observations can be concisely summarized by the following equation

$$\Sigma = \begin{pmatrix} 0 & \Sigma_{AA} & \Sigma_{A\bar{A}} & 0 \\ \Sigma_{BB} & 0 & 0 & \Sigma_{B\bar{B}} \\ \Sigma_{\bar{B}B} & 0 & 0 & \Sigma_{\bar{B}\bar{B}} \\ 0 & \Sigma_{\bar{A}A} & \Sigma_{\bar{A}\bar{A}} & 0 \end{pmatrix}. \quad (96)$$

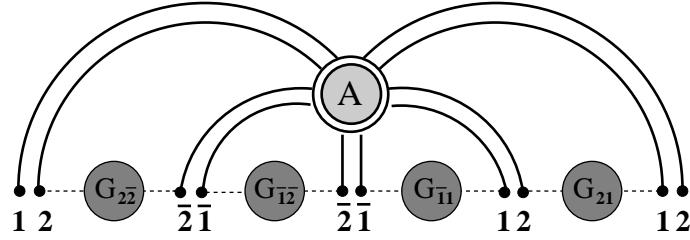


FIG. 9. An example of A -spider connected diagram. Such diagrams are generated by sequences of A and A^\dagger which due to the correspondence (92) $A \leftrightarrow \mathcal{H}_{12}$, $A^\dagger \leftrightarrow \mathcal{H}_{\bar{2}\bar{1}}$ generate sequences of pairs 12 and $\bar{2}\bar{1}$. In this example we have a sequence $12, \bar{2}\bar{1}, \bar{2}\bar{1}, 12, 12$ which begins with the index 1 and ends with 2 . It contributes to $\Sigma_{12} = \mathcal{R}_{12}(\mathcal{G})$, a product of $\mathcal{G}_{2\bar{2}}$, $\mathcal{G}_{\bar{1}2}$, $\mathcal{G}_{\bar{1}\bar{1}}$, \mathcal{G}_{21} which can be read off from the picture by matching the indices on the horizontal line.

The matrix has eight zeros which correspond to (95). The remaining eight elements can be grouped in two groups of four elements each of which can be mapped into a 2×2 matrix. More precisely, the matrix Σ of dimensions 4×4 is expressed in terms of 2×2 generating functions \mathcal{R} and \mathcal{G} for A and B :

$$\Sigma_A = \begin{pmatrix} \Sigma_{AA} & \Sigma_{A\bar{A}} \\ \Sigma_{\bar{A}A} & \Sigma_{\bar{A}\bar{A}} \end{pmatrix} = \begin{pmatrix} \mathcal{R}_{AA}(\mathcal{G}_B) & \mathcal{R}_{A\bar{A}}(\mathcal{G}_B) \\ \mathcal{R}_{\bar{A}A}(\mathcal{G}_B) & \mathcal{R}_{\bar{A}\bar{A}}(\mathcal{G}_B) \end{pmatrix} = \mathcal{R}_A(\mathcal{G}_B) \quad (97)$$

and

$$\Sigma_B = \begin{pmatrix} \Sigma_{BB} & \Sigma_{B\bar{B}} \\ \Sigma_{\bar{B}B} & \Sigma_{\bar{B}\bar{B}} \end{pmatrix} = \begin{pmatrix} \mathcal{R}_{BB}(\mathcal{G}_A) & \mathcal{R}_{B\bar{B}}(\mathcal{G}_A) \\ \mathcal{R}_{\bar{B}B}(\mathcal{G}_A) & \mathcal{R}_{\bar{B}\bar{B}}(\mathcal{G}_A) \end{pmatrix} = \mathcal{R}_B(\mathcal{G}_A) \quad (98)$$

The argument of \mathcal{R}_A in $\Sigma_A = \mathcal{R}_A(\mathcal{G}_B)$ is \mathcal{G}_B while the argument of \mathcal{R}_B in $\Sigma_B = \mathcal{R}_B(\mathcal{G}_A)$ is \mathcal{G}_A as argued above where

$$\mathcal{G}_A = \begin{pmatrix} \mathcal{G}_{12} & \mathcal{G}_{1\bar{1}} \\ \mathcal{G}_{2\bar{2}} & \mathcal{G}_{\bar{2}\bar{1}} \end{pmatrix}, \quad \mathcal{G}_B = \begin{pmatrix} \mathcal{G}_{21} & \mathcal{G}_{2\bar{2}} \\ \mathcal{G}_{\bar{1}\bar{1}} & \mathcal{G}_{\bar{1}2} \end{pmatrix}. \quad (99)$$

So far we have used diagrammatic properties of the equation (88). Now we can also exploit the second equation (87). Inverting the matrix on the left hand side of this equation for the particular form (96) we can find elements of \mathcal{G} as functions of Σ 's. In particular the equation for \mathcal{G}_{11} is

$$\mathcal{G}_{11} = \frac{w\bar{w}^2 - \bar{w}\Sigma_{\bar{A}A}\Sigma_{B\bar{B}} - w\Sigma_{\bar{A}\bar{A}}\Sigma_{\bar{B}\bar{B}}}{\det(\mathcal{W} - \Sigma)} \quad (100)$$

where

$$\begin{aligned} \det(\mathcal{W} - \Sigma) = & w^2\bar{w}^2 - w^2\Sigma_{\bar{A}\bar{A}}\Sigma_{\bar{B}\bar{B}} - \bar{w}^2\Sigma_{AA}\Sigma_{BB} \\ & + (\Sigma_{AA}\Sigma_{\bar{A}\bar{A}} - \Sigma_{A\bar{A}}\Sigma_{\bar{A}\bar{A}})(\Sigma_{B\bar{B}}\Sigma_{\bar{B}\bar{B}} - \Sigma_{BB}\Sigma_{\bar{B}\bar{B}}) \\ & - \bar{w}w(\Sigma_{AA}\Sigma_{\bar{B}B} + \Sigma_{\bar{A}\bar{A}}\Sigma_{BB}) \end{aligned} \quad (101)$$

Now we can use (94) to compare \mathcal{G}_{11}/w that follows from (100) to \mathcal{G}_{MM}

$$\mathcal{G}_{MM} = \frac{\bar{z} - \Sigma_{\bar{M}\bar{M}}}{\det(\mathcal{Z} - \Sigma_M)} \quad (102)$$

where

$$\Sigma_M = (z - \Sigma_{MM})(\bar{z} - \Sigma_{\bar{M}\bar{M}}) - \Sigma_{M\bar{M}}\Sigma_{\bar{M}M} \quad (103)$$

and $z = w^2$. From this comparison we can deduce relations between Σ_M and Σ_A and Σ_B . The numerators of expressions for \mathcal{G}_{11}/w and of \mathcal{G}_{MM} are equal if

$$\Sigma_{\bar{M}\bar{M}} = \frac{\bar{w}}{w}\Sigma_{AA}\Sigma_{B\bar{B}} + \Sigma_{\bar{A}\bar{A}}\Sigma_{BB} \quad (104)$$

and the denominators (101), (103) if

$$\begin{aligned}
(w^2 - \Sigma_{MM})(\bar{w}^2 - \Sigma_{\bar{M}\bar{M}}) - \Sigma_{M\bar{M}}\Sigma_{\bar{M}M} \\
= w^2\bar{w}^2 - w^2\Sigma_{\bar{A}\bar{A}}\Sigma_{\bar{B}\bar{B}} - \bar{w}^2\Sigma_{AA}\Sigma_{BB} \\
+ (\Sigma_{AA}\Sigma_{\bar{A}\bar{A}} - \Sigma_{AA}\Sigma_{\bar{A}\bar{A}})(\Sigma_{B\bar{B}}\Sigma_{\bar{B}B} - \Sigma_{BB}\Sigma_{\bar{B}\bar{B}}) \\
- \bar{w}w(\Sigma_{AA}\Sigma_{\bar{B}\bar{B}} + \Sigma_{\bar{A}\bar{A}}\Sigma_{BB}).
\end{aligned} \tag{105}$$

One can check that the two equations are simultaneously fulfilled if

$$\begin{aligned}
\Sigma_{MM} &= \Sigma_{AA}\Sigma_{BB} + \frac{w}{\bar{w}}\Sigma_{A\bar{A}}\Sigma_{\bar{B}B} \\
\Sigma_{M\bar{M}} &= \sqrt{\frac{\bar{w}}{w}}\Sigma_{AA}\Sigma_{B\bar{B}} + \sqrt{\frac{w}{\bar{w}}}\Sigma_{A\bar{A}}\Sigma_{\bar{B}B} \\
\Sigma_{\bar{M}M} &= \sqrt{\frac{\bar{w}}{w}}\Sigma_{\bar{A}\bar{A}}\Sigma_{BB} + \sqrt{\frac{w}{\bar{w}}}\Sigma_{\bar{A}\bar{A}}\Sigma_{\bar{B}B} \\
\Sigma_{\bar{M}\bar{M}} &= \Sigma_{\bar{A}\bar{A}}\Sigma_{\bar{B}\bar{B}} + \frac{\bar{w}}{w}\Sigma_{\bar{A}\bar{A}}\Sigma_{BB}.
\end{aligned} \tag{106}$$

Remarkably, these equalities can be written in a *factorizable* matrix form as

$$\begin{aligned}
\Sigma_M \equiv \begin{pmatrix} \Sigma_{MM} & \Sigma_{M\bar{M}} \\ \Sigma_{\bar{M}M} & \Sigma_{\bar{M}\bar{M}} \end{pmatrix} &= \begin{pmatrix} \Sigma_{AA} & \sqrt{\frac{w}{\bar{w}}}\Sigma_{A\bar{A}} \\ \sqrt{\frac{\bar{w}}{w}}\Sigma_{\bar{A}A} & \Sigma_{\bar{A}\bar{A}} \end{pmatrix} \cdot \begin{pmatrix} \Sigma_{BB} & \sqrt{\frac{\bar{w}}{w}}\Sigma_{B\bar{B}} \\ \sqrt{\frac{w}{\bar{w}}}\Sigma_{\bar{B}B} & \Sigma_{\bar{B}\bar{B}} \end{pmatrix} \\
&\equiv \Sigma_A^L \Sigma_B^R.
\end{aligned} \tag{107}$$

In order to simplify the notation it is convenient to introduce a 2×2 unitary diagonal matrix U

$$U \equiv \begin{pmatrix} \left(\frac{w}{\bar{w}}\right)^{1/4} & 0 \\ 0 & \left(\frac{\bar{w}}{w}\right)^{1/4} \end{pmatrix} = \begin{pmatrix} e^{+i\frac{\psi}{2}} & 0 \\ 0 & e^{-i\frac{\psi}{2}} \end{pmatrix} \tag{108}$$

where the angle ψ is the phase of w : $w = |w|e^{i\psi}$. Note that w is related to the original variable z as $z = w^2$, so $\text{Arg } z = 2 \text{ Arg } w$. Using this matrix we can associate with any matrix X two similar matrices X^L and X^R obtained by "left and right U -rotations" of the matrix in question

$$X^L \equiv [X]^L = UXU^\dagger, \quad X^R \equiv [X]^R = U^\dagger XU, \tag{109}$$

In particular

$$\Sigma_A^L \equiv [\Sigma_A]^L = U\Sigma_AU^\dagger, \quad \Sigma_A^R \equiv [\Sigma_B]^R = U^\dagger\Sigma_BU. \tag{110}$$

The operations $[\dots]^L$ and $[\dots]^R$ obey simple rules like for instance

$$[XY]^L = [X]^L[Y]^L = X^LY^L, \quad [X^{-1}]^L = ([X]^L)^{-1}, \quad X = [[X]^L]^R. \tag{111}$$

which we will frequently use below.

Now we come to the main result of the paper. Recalling that $\Sigma_A = \mathcal{R}_A(\mathcal{G}_B)$ and $\Sigma_B = \mathcal{R}_B(\mathcal{G}_A)$ we have (107)

$$\mathcal{R}_M(\mathcal{G}_M) = [\mathcal{R}_A(\mathcal{G}_B)]^L \cdot [\mathcal{R}_B(\mathcal{G}_A)]^R. \tag{112}$$

This equation is a cornerstone of the matrix multiplication for non-hermitian matrices. Let us note the similarity with the corresponding equation for the hermitian case (77) albeit with two key differences. Firstly, the objects appearing in (112) are generically noncommuting 2×2 matrices and hence the ordering is crucial. Secondly, the left- and right- U -rotations have no analogue in the scalar hermitian case.

In fact, to arrive to this point we have only taken advantage of the equations for the element \mathcal{G}_{11} of the 4×4 Green's function. Inverting the matrix on the right hand side of (87) for Σ given by (96) we can relate remaining elements of \mathcal{G} to the elements of 2×2 Σ 's and \mathcal{R} 's. In particular we can write equations for elements \mathcal{G}_{12} , $\mathcal{G}_{1\bar{1}}$, $\mathcal{G}_{\bar{2}2}$ and $\mathcal{G}_{\bar{2}\bar{1}}$ which, as we know (99) form a 2×2 matrix corresponding to the Green's function \mathcal{G}_A and similarly for \mathcal{G}_{21} , $\mathcal{G}_{2\bar{2}}$, $\mathcal{G}_{\bar{1}1}$ and $\mathcal{G}_{\bar{1}\bar{2}}$ corresponding to \mathcal{G}_B . This allows us to express \mathcal{G}_A and \mathcal{G}_B in terms of Σ_A and Σ_B . After some straightforward but tedious algebra we arrive at remarkably simple equations which again are analogs of the hermitian equations (78) but with specific ordering and appropriate U -rotations

$$\begin{aligned}
\mathcal{G}_A &= \left[\mathcal{G}_M \cdot [\mathcal{R}_A(\mathcal{G}_B)]^L \right]^L \\
\mathcal{G}_B &= \left[[\mathcal{R}_B(\mathcal{G}_A)]^R \cdot \mathcal{G}_M \right]^R.
\end{aligned} \tag{113}$$

The set of equations (112) and (113) gives the multiplication law for non-hermitian matrices and constitutes one of main results of this work, as mentioned at the beginning of the paper (4).

These equations are in one-to one-correspondence to (77) and (78) except that now instead of complex numbers g_M , g_A and g_B we have 2×2 matrices \mathcal{G}_M , \mathcal{G}_A and \mathcal{G}_B and the additional U -rotations. The logic of the method to calculate the Green's function for the product $M = AB$ is the same as for hermitian matrices, that is for given \mathcal{G}_A and \mathcal{G}_B one determines the matricial R transforms \mathcal{R}_A and \mathcal{R}_B and then applies (112) and (113) to derive the Green's function for M . We will present examples in the next section. Before doing that let us show how these equations can be reformulated in terms of a nonhermitian generalization of the S transform.

D. S transform for non-hermitian matrices

It is natural to anticipate that the S transform for non-hermitian matrices has a form of a 2×2 matrix. It will however appear in two different "left" and "right" versions, since 2×2 matrices do not commute in general. To demonstrate this, we repeat the arguments which have guided us from (77) and (78) to (82), but now we adapt the reasoning to the case of 2×2 matrix-valued transforms.

The first step is to eliminate \mathcal{G}_B and \mathcal{G}_A from the right hand side of (112) and substitute them by \mathcal{G}_M in order to have the same argument on both sides of the equation. To make the following equations slightly more readable we shall skip the subscript M of \mathcal{G}_M writing $\mathcal{G} \equiv \mathcal{G}_M$ and we will denote the inverse matrix of a matrix X as $\frac{1}{X}$ rather than X^{-1} to avoid too many superscripts. Using (113) and (112) we have

$$\mathcal{G}_B = [\mathcal{R}_B^R(\mathcal{G}_A)\mathcal{G}]^R = \left[\frac{1}{\mathcal{R}_A^L(\mathcal{G}_B)} \mathcal{R}_M(\mathcal{G})\mathcal{G} \right]^R. \quad (114)$$

This is an equation for \mathcal{G}_B but \mathcal{G}_B is also present on the right hand side. We can however eliminate \mathcal{G}_B by replacing it recursively by the right hand side and repeating this infinitely many times. In this way we obtain a nested expression (denoted below by dots)

$$\mathcal{G}_B = \left[\frac{1}{\mathcal{R}_A^L(\dots)} \mathcal{R}_M(\mathcal{G})\mathcal{G} \right]^R. \quad (115)$$

that depends on \mathcal{G} and not on \mathcal{G}_B . Thus we can write the first factor, $\mathcal{R}_A^L(\mathcal{G}_B)$, on the right hand side of (112) as a function of \mathcal{G} :

$$\mathcal{R}_A^L(\mathcal{G}_B) = \mathcal{R}_A^L \left(\left[\frac{1}{\mathcal{R}_A^L(\dots)} \mathcal{R}_M(\mathcal{G})\mathcal{G} \right]^R \right) = \frac{1}{SS_A^{(L)}(\mathcal{R}_M(\mathcal{G})\mathcal{G})} \quad (116)$$

where $SS^{(L)}$ is a left S transform defined as

$$SS^{(L)}(\mathcal{Y}) = \frac{1}{\mathcal{R}^L \left(\left[\frac{1}{\mathcal{R}^L(\dots)} \mathcal{Y} \right]^R \right)}. \quad (117)$$

Let us make two further remarks concerning the notation. In the last equation we skipped the subscript A of SS and \mathcal{R} since the relation is valid for any matrix. The superscript (L) of SS is used on purpose in parentheses to distinguish it from L and to emphasize that the left S transform is not a left rotation of the S transform $SS^{(L)} \neq [SS]^L \equiv USU^\dagger$ in contrast to the notation $\mathcal{R}^L = [\mathcal{R}]^L$. The function $SS^{(L)}$ is just defined by the equation above. This equation is equivalent to

$$SS^{(L)}(\mathcal{Y}) = \frac{1}{\mathcal{R}^L \left([SS^{(L)}(\mathcal{Y})\mathcal{Y}]^R \right)} \quad (118)$$

and

$$\mathcal{R}^L(\mathcal{Y}) = \frac{1}{SS^{(L)} \left([\mathcal{R}(\mathcal{Y})\mathcal{Y}]^L \right)}. \quad (119)$$

in analogy to the hermitian case discussed in section V A. Now we can repeat all the steps for the second factor on the right hand side of (112). The result can be written using a right S transform, which is given by two equivalent, reciprocal, equations analogous to those of the left S transform above:

$$SS^{(R)}(\mathcal{Y}) = \frac{1}{\mathcal{R}^R \left([\mathcal{Y}SS^{(R)}(\mathcal{Y})]^L \right)} \quad (120)$$

or

$$\mathcal{R}^R(\mathcal{Y}) = \frac{1}{SS^{(R)}([\mathcal{Y}\mathcal{R}(\mathcal{Y})]^R)} . \quad (121)$$

Using the left and right S transforms we can write (112) in a concise form

$$\frac{1}{\mathcal{R}_M(\mathcal{G})} = SS_B^{(R)}(\mathcal{G}\mathcal{R}_M(\mathcal{G})) \cdot SS_A^{(L)}(\mathcal{R}_M(\mathcal{G})\mathcal{G}) \quad (122)$$

that depends on \mathcal{G} on both sides. This is an equation for the R transform $\mathcal{R}_M(\mathcal{G})$ which in turn determines the generalized Green's function giving the eigenvalue density.

Let us rewrite now the left-hand side using either equation (119) or (121)

$$\left[SS_M^{(L)}([\mathcal{R}(\mathcal{G})\mathcal{G}]^L) \right]^R = \left[SS_M^{(R)}([\mathcal{G}\mathcal{R}(\mathcal{G})]^R) \right]^L = SS_B^{(R)}(\mathcal{G}\mathcal{R}_M(\mathcal{G})) \cdot SS_A^{(L)}(\mathcal{R}_M(\mathcal{G})\mathcal{G}) \quad (123)$$

which now (almost) takes the form of a multiplication law for SS transforms with the only subtlety being the noncommutativity of the arguments.

In the special case when \mathcal{G} and $\mathcal{R}(\mathcal{G})$ commute

$$[\mathcal{G}, \mathcal{R}_M(\mathcal{G})] = 0 \quad (124)$$

we get a direct analogue of the hermitian multiplication law for S transforms since all functions are evaluated on the same argument $\mathcal{Y} = \mathcal{G}\mathcal{R}(\mathcal{G}) = \mathcal{R}(\mathcal{G})\mathcal{G}$. In this case it would make sense to introduce yet another S transform:

$$\mathcal{R}(\mathcal{Y}) = \frac{1}{SS(\mathcal{R}(\mathcal{Y})\mathcal{Y})} , \quad (125)$$

which does not involve any left or right U -rotation. It is easy to see that in this case the equation (122) can be rewritten as

$$SS(\mathcal{Y}) = SS_B^{(R)}(\mathcal{Y}) \cdot SS_A^{(L)}(\mathcal{Y}) . \quad (126)$$

One should note that the 2×2 formalism, which has been developed here for non-hermitian random matrix ensembles, contains also the standard hermitian case. For hermitian matrices, namely, the Green's functions and the R transforms reduce to diagonal matrices

$$\mathcal{G}(\mathcal{Z}) = \begin{pmatrix} G(z) & 0 \\ 0 & \bar{G}(z) \end{pmatrix} , \quad \mathcal{R}(\mathcal{G}) = \begin{pmatrix} R(G) & 0 \\ 0 & \bar{R}(G) \end{pmatrix} , \quad (127)$$

Moreover $\mathcal{G} = \mathcal{G}^L = \mathcal{G}^R$, $\mathcal{R} = \mathcal{R}^L = \mathcal{R}^R$ because the matrix U that defines the left and right rotations (109) is diagonal too and the product of the diagonal elements gives one. It follows also that $SS = SS^{(R)} = SS^{(L)}$ and that the S transform is diagonal $SS(\mathcal{Z}) = \text{diag}(S(z), \bar{S}(z))$ too. Therefore in this case (122) takes a diagonal from

$$\begin{pmatrix} S_M(z) & 0 \\ 0 & \bar{S}_M(z) \end{pmatrix} = \begin{pmatrix} S_A(z)S_B(z) & 0 \\ 0 & \bar{S}_A(z)\bar{S}_B(z) \end{pmatrix} \quad (128)$$

that is equivalent to (82).

VI. EXAMPLES

In this section we will illustrate our methods by presenting three examples. We will start from two examples which cannot be treated, even in the hermitian case, by the conventional S transform treatment as *both* of the random matrix factors of the product are centered. Finally we treat a more complicated example of obtaining a nontrivial two-dimensional eigenvalue distribution for a product of two simple factors.

A. Product of two free Ginibre-Girko matrices

Let us first consider the product $M = AB$ of two identically distributed free Ginibre-Girko matrices: A and B . Throughout this section we will parametrize matrix elements of the 2×2 Green's functions (30) with two complex functions $a = a(z, \bar{z})$ and $b = b(z, \bar{z})$

$$\mathcal{G} = \begin{pmatrix} \mathcal{G}_{11} & \mathcal{G}_{1\bar{1}} \\ \mathcal{G}_{\bar{1}1} & \mathcal{G}_{\bar{1}\bar{1}} \end{pmatrix} = \begin{pmatrix} a & ib \\ i\bar{b} & \bar{a} \end{pmatrix} \quad (129)$$

The R transform for a Ginibre-Girko matrix reads (47).

$$\mathcal{R}(\mathcal{G}) = \mathcal{R} \left(\begin{pmatrix} a & ib \\ i\bar{b} & \bar{a} \end{pmatrix} \right) = \begin{pmatrix} 0 & ib \\ i\bar{b} & 0 \end{pmatrix} \quad (130)$$

and its left and right versions

$$\mathcal{R}^L(\mathcal{G}) = \begin{pmatrix} 0 & \sqrt{\frac{\bar{w}}{w}}ib \\ \sqrt{\frac{w}{\bar{w}}}i\bar{b} & 0 \end{pmatrix}, \quad \mathcal{R}^R(\mathcal{G}) = \begin{pmatrix} 0 & \sqrt{\frac{w}{\bar{w}}}ib \\ \sqrt{\frac{\bar{w}}{w}}i\bar{b} & 0 \end{pmatrix} \quad (131)$$

respectively. We recall that w is related to z as $z = w^2$. Let us now apply the multiplication law for $M = AB$ where A and B are Ginibre-Girko matrices with unit variance. Using (112) we have

$$\mathcal{R}_M = \begin{pmatrix} 0 & \sqrt{\frac{\bar{w}}{w}}ib_B \\ \sqrt{\frac{w}{\bar{w}}}i\bar{b}_B & 0 \end{pmatrix} \begin{pmatrix} 0 & \sqrt{\frac{\bar{w}}{w}}ib_A \\ \sqrt{\frac{w}{\bar{w}}}i\bar{b}_A & 0 \end{pmatrix} = \begin{pmatrix} -\frac{w}{\bar{w}}b_B\bar{b}_A & 0 \\ 0 & -\frac{\bar{w}}{w}\bar{b}_Bb_A \end{pmatrix} \quad (132)$$

Since both A and B are identically distributed they have identical Green's function, we can thus reduce the problem by introducing a single function $b = b_A = b_B$:

$$\mathcal{R}_M = \begin{pmatrix} -\frac{w}{\bar{w}}|b|^2 & 0 \\ 0 & -\frac{\bar{w}}{w}|b|^2 \end{pmatrix} \quad (133)$$

We can now use the two remaining equations of the multiplication law (113) which can be conveniently written as

$$\begin{aligned} \mathcal{G}_M^{-1} [\mathcal{G}_A]^R &= [\mathcal{R}_A(\mathcal{G}_B)]^L \\ [\mathcal{G}_B]^L \mathcal{G}_M^{-1} &= [\mathcal{R}_B(\mathcal{G}_A)]^R \end{aligned} \quad (134)$$

In case of identically distributed A and B one of the two equations is redundant and thus it is sufficient to use only one of them, for instance the first one. We first eliminate \mathcal{G}_M from this equation by using the relation $\mathcal{G}_M^{-1} = \mathcal{Z} - \mathcal{R}_M$ with \mathcal{R}_M given by (133):

$$(\mathcal{Z} - \mathcal{R}_M) [\mathcal{G}_A]^R = [\mathcal{R}_A(\mathcal{G}_B)]^L. \quad (135)$$

This is an explicit equation for a and b

$$\begin{pmatrix} w^2 + \frac{w}{\bar{w}}|b|^2 & 0 \\ 0 & \bar{w}^2 + \frac{\bar{w}}{w}|b|^2 \end{pmatrix} \begin{pmatrix} a & \sqrt{\frac{\bar{w}}{w}}ib \\ \sqrt{\frac{w}{\bar{w}}}i\bar{b} & \bar{a} \end{pmatrix} = \begin{pmatrix} 0 & \sqrt{\frac{\bar{w}}{w}}ib \\ \sqrt{\frac{w}{\bar{w}}}i\bar{b} & 0 \end{pmatrix} \quad (136)$$

It can be easily solved. It has two solutions: a trivial and $a = b = 0$ and a non-trivial one $a = 0$, $|b|^2 = 1 - w\bar{w}$. The latter one is equivalent to $a = 0$ and $|b|^2 = 1 - \sqrt{z\bar{z}}$ when expressed in the variable $z = w^2$. This solution holds inside the unit circle: $z\bar{z} \leq 1$ on the z complex plane while the trivial one outside. The boundary of the eigenvalue distribution in the z plane is given by the condition $b = 0$ for the non-trivial solution which leads to the unit circle. Inserting these solutions to (133) and calculating \mathcal{G}_M we find

$$\mathcal{G}_M(z, \bar{z}) = \begin{pmatrix} \sqrt{\frac{\bar{z}}{z}} & 0 \\ 0 & \sqrt{\frac{z}{\bar{z}}} \end{pmatrix}, \quad \text{for } |z| \leq 1 \quad (137)$$

or

$$\mathcal{G}_M(z, \bar{z}) = \begin{pmatrix} z^{-1} & 0 \\ 0 & \bar{z}^{-1} \end{pmatrix}, \quad \text{for } |z| \geq 1 \quad (138)$$

from which we obtain a rotationally symmetric eigenvalue density for $|z| < 1$:

$$\rho(x, y) = \frac{1}{\pi} \frac{\partial}{\partial \bar{z}} G(z, \bar{z}) = \frac{1}{2\pi} \frac{1}{|z|} \quad (139)$$

inside the unit circle and $\rho(x, y) = 0$ outside. We remind the reader that $G(z, \bar{z})$ is equal to the upper left element \mathcal{G}_{MM} of \mathcal{G}_M .

B. Product of two free GUE matrices

We would like to discuss a simple but very interesting case of the product $M = AB$ of two matrices from the Gaussian Unitary Ensembles. Both A and B are hermitian but their product is not. Since both matrices have a vanishing mean the traditional use of S transform leads to contradiction, as shown in [18]. However, our algorithm works in this case without any problems.

Before we apply the full non-hermitian version of the multiplication law let us check what happens if one applies its hermitian version given by equations (77) and (78). One can do this since A and B are hermitian. However the result for $G_M(z)$ can be interpreted only as a moments' generating function but not as a full Green's function. In particular one cannot use it to reconstruct the eigenvalue density (8) since the eigenvalues are not constrained to the real axis.

For a standardised GUE matrix we have $R(z) = z$ and thus the multiplication law (77) and (78) simplifies to

$$R_M(g) = g_B g_A, \quad g_A = g g_B, \quad g_B = g g_A \quad (140)$$

The two latter relations yield an equation $g_A = g^2 g_A$. Its solution is $g_A = 0$ giving $R_M(g) = 0$ and hence

$$G_M(z) = \frac{1}{z} \quad (141)$$

The moments m_k are given by coefficients at $1/z^{k+1}$ of the $1/z$ -expansion of $G_M(z)$. We see that all they vanish except the trivial one $m_0 = \frac{1}{N} \langle \text{tr} M^0 \rangle = 1$. Of course it does not mean that all eigenvalues of M vanish. In order to determine the eigenvalue density of M one has to apply the full multiplication law in the domain of non-hermitian matrices (112) and (113). The calculation goes along the same lines as in the previous example except that now instead of (130) the R transform is

$$\mathcal{R}(\mathcal{G}) = \mathcal{R} \left(\begin{pmatrix} a & ib \\ i\bar{b} & \bar{a} \end{pmatrix} \right) = \begin{pmatrix} a & ib \\ i\bar{b} & \bar{a} \end{pmatrix} = \mathcal{G} \quad (142)$$

as follows from (46) for $\tau = 1$. It is easy to see that the solution is exactly the same as in the previous example since for $a = 0$ (which was a solution) all equations reduce to those for the previous case. This result is in agreement with the recent works [19, 20, 22]. Actually one can see that the same holds for any elliptic ensemble with

$$\mathcal{R}(\mathcal{G}) = \mathcal{R} \left(\begin{pmatrix} \tau a & ib \\ i\bar{b} & \tau \bar{a} \end{pmatrix} \right) = \begin{pmatrix} \tau a & ib \\ i\bar{b} & \tau \bar{a} \end{pmatrix} = \mathcal{G} \quad (143)$$

since again for $a = 0$ the equations are identical as before. Again this is in agreement with [19] where it was shown that even for A and B being from different elliptic ensembles ($\tau_A \neq \tau_B$ or $\tau_A = \tau_B$) one obtains the same circular law (139).

C. Pascal limaçon

We shall calculate now the eigenvalue distribution of the product of two shifted Ginibre-Girko matrices $M = AB = (1 + X_A)(1 + X_B)$ where X_A and X_B are free Ginibre-Girko complex matrices. The main difference to the cases discussed before is that the multiplied matrices A and B are not centered: $\frac{1}{N} \text{tr} A = 1$ and $\frac{1}{N} \text{tr} B = 1$, so their first moments (cumulant) are not zero:

$$\begin{aligned} \mathcal{R}_A(\mathcal{G}_B) &= \mathcal{R}_A \left(\begin{pmatrix} a_B & ib_B \\ i\bar{b}_B & \bar{a}_B \end{pmatrix} \right) = \begin{pmatrix} 1 & ib_B \\ i\bar{b}_B & 1 \end{pmatrix} \\ \mathcal{R}_B(\mathcal{G}_A) &= \mathcal{R}_B \left(\begin{pmatrix} a_A & ib_A \\ i\bar{b}_A & \bar{a}_A \end{pmatrix} \right) = \begin{pmatrix} 1 & ib_A \\ i\bar{b}_A & 1 \end{pmatrix} \end{aligned} \quad (144)$$

Since A and B are identically distributed we set $b = b_A = b_B$ as in the previous examples. Using (112) we have

$$\mathcal{R}_M = \begin{pmatrix} 1 - \frac{w}{\bar{w}} |b|^2 & ib \left(\sqrt{\frac{\bar{w}}{w}} + \sqrt{\frac{w}{\bar{w}}} \right) \\ i\bar{b} \left(\sqrt{\frac{\bar{w}}{w}} + \sqrt{\frac{w}{\bar{w}}} \right) & 1 - \frac{\bar{w}}{w} |b|^2 \end{pmatrix} \quad (145)$$

Inserting this to (135) we obtain an explicit equation

$$\begin{pmatrix} w^2 - 1 + \frac{w}{\bar{w}}|b|^2 & -ib\left(\sqrt{\frac{\bar{w}}{w}} + \sqrt{\frac{w}{\bar{w}}}\right) \\ -i\bar{b}\left(\sqrt{\frac{\bar{w}}{w}} + \sqrt{\frac{w}{\bar{w}}}\right) & \bar{w}^2 - 1 + \frac{\bar{w}}{w}|b|^2 \end{pmatrix} \begin{pmatrix} a & \sqrt{\frac{\bar{w}}{w}}ib \\ \sqrt{\frac{w}{\bar{w}}}i\bar{b} & \bar{a} \end{pmatrix} = \begin{pmatrix} 1 & \sqrt{\frac{\bar{w}}{w}}ib \\ \sqrt{\frac{\bar{w}}{w}}i\bar{b} & 1 \end{pmatrix} \quad (146)$$

which reduces to two equations for a and $|b|^2$:

$$\begin{aligned} a(w^2 - 1 + |b|^2 \frac{w}{\bar{w}}) + |b|^2(1 + \frac{w}{\bar{w}}) &= 1 \\ -a(1 + \frac{w}{\bar{w}}) + \frac{w}{\bar{w}}(\bar{w}^2 - 1) + |b|^2 &= 1 \end{aligned} \quad (147)$$

This set of equations has a trivial solution: $b = 0$ and $a = 1/(w^2 - 1)$ and a non-trivial one that can be found by eliminating a from the last set of equations. This gives an equation for $C = |b|^2$ (41):

$$C^2 + C(1 + 2|w|^2) + |w|^4 - |w|^2 - \bar{w}^2 - w^2 = 0 \quad (148)$$

The border line between the two solutions can be found by setting $C = 0$ in the last equation [21]:

$$\bar{w}^2 w^2 - \bar{w}w = w^2 + \bar{w}^2 \quad (149)$$

It represents a curve on the z -plane called Pascal's limaçon after Etienne Pascal (1588-1651) - the father of Blaise Pascal. It has a more familiar form in polar coordinates on the z -plane: $w^2 \equiv z = r \exp i\phi$:

$$r = 1 + 2 \cos \phi. \quad (150)$$

It is a particular case of the trisectrix. The trivial solution holds outside the Pascal's limaçon while the non-trivial inside. For the trivial solution the Green's function is $G = \mathcal{G}_{MM} = 1/(z - 1)$ and thus $\rho(x, y) = 0$. The non-trivial solution can be found by inverting $\mathcal{G}_M = (\mathcal{Z} - \mathcal{R}_M)^{-1}$ for \mathcal{R}_M (145). The Green's function is given by the upper left element of \mathcal{G}_M

$$G = \mathcal{G}_{MM} = \frac{\bar{w}^2 - 1 + \frac{\bar{w}}{w}C}{(w^2 - 1 + \frac{w}{\bar{w}}C)(\bar{w}^2 - 1 + \frac{\bar{w}}{w}C) + C\left(\frac{\bar{w}}{w} + 2 + \frac{w}{\bar{w}}\right)} \quad (151)$$

with C being a solution of (148), and again agrees with [21]. One can write down the solution in polar coordinates on the z -plane: $z = re^{i\phi}$ as

$$G \equiv G_x - iG_y = \frac{(r + C) \cos \phi - 1}{D} - i \frac{(r + C) \sin \phi}{D} \quad (152)$$

where C and D are real non-negative functions

$$C = \frac{1}{2} \left(-1 - 2r + \sqrt{1 + 8r(1 + \cos \phi)} \right) \quad (153)$$

and

$$D = ((r + C) \cos \phi - 1)^2 + (r + C)^2 \sin^2 \phi + 2C(1 + \cos \phi) \quad (154)$$

The first one corresponds to (148) and the second one to the denominator in (151). One can explicitly see that C is positive inside the Pascal limaçon $r < 1 + 2 \cos \phi$. Using the Gauss law we find the eigenvalue density

$$\rho = \frac{1}{\pi} \frac{\partial G}{\partial \bar{z}} = \frac{1}{2\pi} \left(\frac{\partial G_x}{\partial x} + \frac{\partial G_y}{\partial y} \right) = \frac{1}{2\pi} \operatorname{div} \vec{G}. \quad (155)$$

The imaginary part of $\partial_{\bar{z}} G$ is proportional to the rotation $\operatorname{rot} \vec{G} = \partial_x G_y - \partial_y G_x$ that vanishes by construction. This fact can be used as a test of correctness of calculations. The density calculated from this formula is shown in figure 10. Finally, we perform some numerical checks. We generate numerically matrices $M = AB = (1 + X_A)(1 + X_B)$ of dimensions 100×100 and compare obtained eigenvalue histograms with the exact solution for infinite dimensions. In figure 11 we show a scattered plot of eigenvalues and the histogram of real eigenvalues compared to the section of the analytic solution along the real axis. The results show a good agreement between numerical data and the analytic result. The small remaining deviations can be attributed to finite size effects.

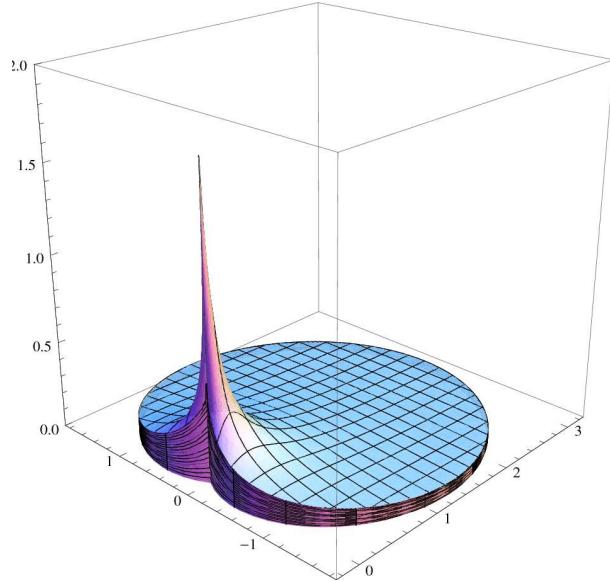


FIG. 10. The eigenvalue density of the product of two shifted Ginibre-Girko matrices. It is non-zero in the region $r \leq 1 + 2 \cos \phi$. The density is peaked around the origin. The maximum of the function is located at the origin: $\rho(x = 0, y = 0) = \frac{6}{\pi} \approx 1.90986$ while the minimum at the point $x = 3, y = 0$: $\rho(x = 3, y = 0) = \frac{9\pi}{56} \approx 0.0511569$.

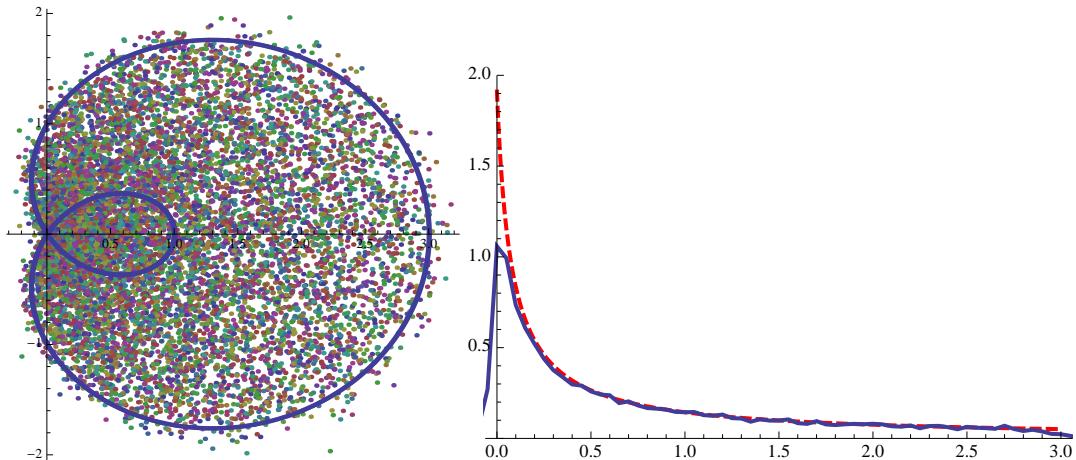


FIG. 11. (Left) The analytical contour $r = 1 + 2 \cos \phi$ (150) and the scattered plot of eigenvalues obtained by diagonalisation of 100 matrices of dimensions 100×100 . One should note that the boundary of the support is formed only by the external part of the Pascal's limaçon which corresponds to $\phi \in [-2\pi/3, 2\pi/3]$. The remaining part of the limaçon lies inside the support. (Right) A numerical histogram (solid line) constructed from almost real eigenvalues (whose imaginary part is less than $\epsilon = 10^{-2}$) obtained by diagonalisation of 20000 matrices of size 100×100 compared to the section of the analytic eigenvalue density ρ along the real axis. The deviations between the numerical histogram and the theoretical curve are caused by finite size effects.

VII. SUMMARY

We have introduced a natural generalization of the concept of S transform for the product of non-hermitian ensembles. This construction puts on the same footing addition and multiplication laws for hermitian and non-hermitian ensembles. We have also found a more general reformulation of the multiplication law which allows us to calculate free products of random matrices having vanishing mean, including the case when both factors in the product are centered. This case is especially interesting as it cannot be addressed using ordinary S transform techniques.

Our construction relies on the insights from diagrammatic techniques, and in particular assumes the finiteness of the moments. We are however convinced, that these conditions are neither restrictive nor mandatory for a general proof, based on purely algebraic structures like e.g. amalgamation of free random variables and a careful treatment

of regularization of ensembles with ubounded moments.

Acknowledgments

This work was partially supported by the Polish Ministry of Science Grants No. N N202 22913 and N N202 105136. ZB and MAN would like to thank the Nordita at Stockholm, where a part of this work has been completed, for the hospitality. Their stay at Nordita was supported by the program "Random Geometry and Applications".

[1] D. Voiculescu, *Invent. Math.* **104** (1991) 201; D.V. Voiculescu, K.J. Dykema and A. Nica, *Free Random Variables*, (Am. Math. Soc., Providence, RI, 1992).

[2] R. Speicher, *Math. Ann.* **298** (1994) 611.

[3] D. Tse and O. Zeitouni, *IEEE Trans. Inform. Theory* **46** 172 (2000).

[4] I.E. Telatar, *Eur. Trans. Telecommun.* **10** 585 (1999).

[5] R.R. Müller, *IEEE Trans. Inform. Theory* **48** 2495 (2002).

[6] A. Tulino and S. Verdú, *Jour. Commun. and Inform. Theory* **bf 1** (2004) 1.

[7] Z. Burda, A. Jarosz, M.A. Nowak, J. Jurkiewicz, G. Papp and I. Zahed, *Quantitative Finance*, Dec.21 (2010) SN-1469-7688.

[8] J.-P. Bouchaud and M. Potters, *Quantitative Finance Papers*, 0910.1205.

[9] R.A. Janik, M.A. Nowak, G. Papp and I. Zahed, *Acta Phys. Polon.* **B28** (1997) 2949 and references therein.

[10] M. Debbah and O. Ryan, *IEEE trans. Signal Process.* **56** (2008) 5654.

[11] R.A. Janik, M.A. Nowak, G. Papp, J. Wambach and I. Zahed, *Phys. Rev.* **E55** (1997) 4100.

[12] R.A. Janik, M.A. Nowak, G. Papp and I. Zahed, *Nucl. Phys.* **B501** (1997) 603.

[13] J. Feinberg and A. Zee, *Nucl. Phys.* **B501** (1997) 643; J. Feinberg and A. Zee, *Nucl. Phys.* **B504** (1997) 579.

[14] J.T. Chalker and Z. Jane Wang, *Phys. Rev. Lett.* **79** (1997) 1797.

[15] A. Jarosz and M.A. Nowak, *J. Phys.* **A39** (2006) 10107.

[16] T. Rodgers, *J. Math. Phys.* **51** (2010) 093304.

[17] P. Zinn-Justin, *Phys. Rev.* **E59** (1999) 4884.

[18] R. Speicher and N.R. Rao, *Elect. Comm. in Probab.* **12** (2007) 248.

[19] Z. Burda, R.A. Janik and B. Wacław, *Phys. Rev.* **E81** (2010) 041132.

[20] Z. Burda, A. Jarosz, G. Livan, M.A. Nowak and A. Święch, *Phys. Rev.* E82 (2010) 061114.

[21] E. Gudowska-Nowak, R.A. Janik, J. Jurkiewicz and M.A. Nowak, *Nucl. Phys.* **B670** (2003) 479.

[22] V.L. Girko and A.I. Vladimirova, *Random operators and stochastic equations*, **17(3)** (2009) 247.

[23] E. Wigner, *Can. Math. Congr. Proc.* p.174 (University of Toronto Press) and other papers reprinted in C.E. Porter *Statistical Theories of Spectra: Fluctuations* (Academic Press, New York, 1965);
M.L. Mehta, *Random Matrices* (Academic Press, New York, 1991).

[24] V.L. Girko, *Spectral theory of random matrices* (in Russian), Nauka, Moscow (1988) and references therein.

[25] F. Haake et al., *Zeit. Phys.* **B88** (1992) 359;
N. Lehmann, D. Saher, V.V. Sokolov and H.-J. Sommers, *Nucl. Phys.* **A582** (1995) 223.

[26] Y.V. Fyodorov and H.-J. Sommers, *J. Math. Phys.* **38** (1997) 1918;
Y.V. Fyodorov, B.A. Khoruzhenko and H.-J. Sommers, *Phys. Lett.* **A226** (1997) 46;
H.-J. Sommers, A. Crisanti, H. Sompolinsky and Y. Stein, *Phys. Rev. Lett.* **60** (1988) 1895.

[27] R.A. Janik, W. Noerenberg, M.A. Nowak, G. Papp and I. Zahed, *Phys. Rev.* **E60** (1999) 2699.

[28] J.T. Chalker and B. Mehlig, *Phys. Rev. Lett.* **81** (1998) 3367.

[29] D.V. Voiculescu, *Multiplication of certain non-commuting random variables*, *J. Operator Theory* **18** (1987) 223-235.

[30] J. Ginibre, *J. Math. Phys.* **6** (1965) 440.

[31] R.A. Janik, Ph.D. Thesis, Cracow 1996, (unpublished).